Construction of a gyrogroup from a group

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Outline

What is a gyrogroup?

Gyrogroup—group-like structure

- Consisting of one set with one binary operation
- Operation NOT associative, NOT a group, in general
- Having associativity-correction maps—gyroautomorphisms
- Having algebraic properties like groups
- Being a generalization of groups
- First introduced by Abraham A. Ungar

Gyrogroups—an axiom approach

Let G be a non-empty set and let $\oplus$ be a binary operation on G. Then $(G, \oplus)$ is a gyrogroup if

1. $\exists e \in G\ \forall a \in G,\ a \oplus e = a = e \oplus a$
2. $\forall a \in G \exists b \in G,\ b \oplus a = e = a \oplus b$
3. $\forall a, b \in G\ \exists \mathrm{gyr}[a, b], \mathrm{gyr}[b, a] \in \mathrm{Aut}\,(G, \oplus)$ such that
   - $a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b]c$ (left gyroassociative law)
   - $(a \oplus b) \oplus c = a \oplus (b \oplus \mathrm{gyr}[b, a]c)$ (right gyroassociative law)
4. $\forall a, b \in G$,
   - $\mathrm{gyr}[a \oplus b, b] = \mathrm{gyr}[a, b]$ (left loop property)
   - $\mathrm{gyr}[a, b \oplus a] = \mathrm{gyr}[a, b]$ (right loop property)

Gyrocommutative gyrogroups

A gyrogroup $(G, \oplus)$ that satisfies the commutative-like law,

$$a \oplus b = \mathrm{gyr}[a, b](b \oplus a) \tag{1}$$

for all $a, b \in G$, is called a gyrocommutative gyrogroup, analogous to abelian groups.

Concrete example of a gyrogroup—Möbius addition

Set $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$. Möbius addition [1], $\oplus_M$, is given by

$$a \oplus_M b = \frac{a + b}{1 + \bar{a}b} \tag{2}$$

for all $a, b \in \mathbb{D}$. Then $(\mathbb{D}, \oplus_M)$ forms a gyrocommutative gyrogroup that is not a group.

[1] A. Ungar, *The holomorphic automorphism group of the complex disk*, Aequationes Mathmematicae **47** (1994)

Groups and gyrogroups

Recall the gyroassociative law

$$\begin{aligned} a \oplus (b \oplus c) &= (a \oplus b) \oplus \mathrm{gyr}[a, b]c \\ (a \oplus b) \oplus c &= a \oplus (b \oplus \mathrm{gyr}[b, a]c) \end{aligned}$$

- Every group is a gyrogroup by defining $\mathrm{gyr}[a, b]$ to be the identity automorphism.
- Any gyrogroup with trivial gyroautomorphisms is a group.

A non-degenerate gyrogroup is a gyrogrop that has at least one non-trivial gyroautomorphism.

Groups and gyrogroups

GROUP	GYROGROUP
group identity 1	gyrogroup identity e
inverse element a^{-1}	inverse element $\ominus a$
the associative law	the gyroassociative law
subgroup	subgyrogroup
normal subgroup	normal subgyrogroup
quotient group	quotient gyrogroup
group homomorphism	gyrogroup homomorphism
group isomorphism	gyrogroup isomorphism
abelian group	gyrocommutative gyrogroup
$\vdots$	$\vdots$

[2] T. S., Essays in Mathematics and Its Applications: In Honor of Vladimir Arnold, in: Th.M. Rassias, P.M. Pardalos (Eds.), *The Algebra of Gyrogroups: Cayley's Theorem, Lagrange's Theorem, and Isomorphism Theorems*, Springer, Cham, 2016, pp.369–437

Relationship between a gyrogroup and its symmetric group

Let G be a gyrogroup and let $a \in G$. The left gyrotranslation by a, denoted by L_a and defined by $L_a(x) = a \oplus x, x \in G$, is a permutation of G. Set

$$\hat{G} = \{L_a \colon a \in G\}.$$

Here is a nice relationship between $\hat{G}$ and $\mathrm{Sym}(G)$.

Theorem 1

Viewing $\mathrm{Sym}(G)$ as the usual symmetric group, we have

1. L_e, which is the identity map, is in $\hat{G}$
2. $X \in \hat{G}$ implies $X^{-1} \in \hat{G}$
3. $X, Y \in \hat{G}$ implies $X \circ Y \circ X \in \hat{G}$.

That is, $\hat{G}$ is a twisted subgroup, but not subgroup, of $\mathrm{Sym}(G)$.

Gyrotriples

A subset B of a group Γ is a twisted subgroup of Γ if (i) $1 \in B$, 1 being the identity of Γ; (ii) $b \in B$ implies $b^{-1} \in B$; and (iii) $a, b \in B$ implies $aba \in B$.

A subset B of a group Γ is a (left) transversal to a subgroup Ξ of Γ if each element g of Γ can be written uniquely as $g = bh$ for some $b \in B$ and $h \in \Xi$.

Definition 2

Let Γ be a group, let B be a subset of Γ, and let Ξ be a subgroup of Γ. A triple (Γ, B, Ξ) is called a gyrotriple if the following properties hold:

1. B is a transversal to Ξ in Γ
2. B is a twisted subgroup of Γ
3. Ξ normalizes B, that is, $hBh^{-1} \subseteq B$ for all $h \in \Xi$.

Gyrogroup → Group

Let G be a gyrogroup. Then $\Sigma = \{L_a \circ \alpha \colon a \in G, \alpha \in \mathrm{Aut}(G)\}$ forms a group under composition of maps with group law:

$$(L_a \circ \alpha) \circ (L_b \circ \beta) = L_{a \oplus \alpha(b)} \circ (\mathrm{gyr}[a, \alpha(b)] \circ \alpha \circ \beta) \tag{3}$$

for all $a, b \in G, \alpha, \beta \in \mathrm{Aut}(G)$. Furthermore, $\hat{G} \subseteq \Sigma$ and $\mathrm{Aut}(G)$ is a subgroup of Σ.

Theorem 3 (T. S., 2017)

If G is a gyrogroup, then $(\Sigma, \hat{G}, \mathrm{Aut}(G))$ is a gyrotriple.

Group → Gyrogroup

Suppose that a subset B of a group Γ is a transversal to a subgroup Ξ of a group Γ. By definition, for all $a, b \in B$, there are unique elements $a \odot b \in B$ and $h(a, b) \in \Xi$ such that $ab = (a \odot b)h(a, b)$. In some case, $\odot$ becomes a gyrogroup operation.

Theorem 4 (T. Foguel & A. Ungar, 2000 • T. S., 2017)

Let (Γ, B, Ξ) be a gyrotriple. Then B is a gyrogroup under the transversal operation. For all $a, b \in B$, the gyroautomorphism of B generated by a and b is conjugation by $h(a, b)$.

In this case, the group identity of Γ acts as the gyrogroup identity of B and $\ominus b = b^{-1}$ for all $b \in B$.

Involutive groups

A group Γ, together with an automorphism τ of Γ such that $\tau^2 = I_\Gamma$, is called an involutive group [3], denoted by (Γ, τ). In this case, τ induces a *subset* $G(\Gamma)$ and a *subgroup* $A(\Gamma)$ of Γ given by

$$G(\Gamma) = \{gg^\dagger : g \in \Gamma\} \quad \text{and} \quad A(\Gamma) = \{g \in \Gamma : \tau(g) = g\}. \tag{4}$$

Here, $g^\dagger = \tau(g)^{-1}$ and the map $\dagger$ defines an involutive anti-automorphism of Γ.

Proposition 5

If (Γ, τ) is an involutive group, then $G(\Gamma)$ is a twisted subgroup of Γ and $A(\Gamma)$ normalizes $G(\Gamma)$.

[3] J. Lawson, Clifford algebras, Möbius transformations, Vahlen matrices, and B-loops, *Comment. Math. Univ. Carolin.* **51**(2010), no. 2, pp. 319–331

Construction of a gyrogroup I

A subset B of a group Γ is uniquely 2-divisible if for each element a of B, there is a unique element b of B such that $b^2 = a$. In this case, $\sqrt{a}$ denotes the unique element of B such that $\sqrt{a}^2 = a$.

Theorem 6 (T. S., 2017)

Let (Γ, τ) be an involutive group. If $G(\Gamma)$ is uniquely 2-divisible, then $(\Gamma, G(\Gamma), A(\Gamma))$ is a gyrotriple. In this case, $G(\Gamma)$ forms a gyrogroup under the operation given by

$$a \oplus b = \sqrt{ab^2a}, \tag{5}$$

where the gyroautomorphisms of $G(\Gamma)$ are given by

$$\text{gyr}[a, b]c = hch^{-1}, \qquad h = \sqrt{ab^2a}^{-1}ab, \tag{6}$$

for all $a, b, c \in G(\Gamma)$.

Concrete example—real matrices

Let $\mathrm{GL}_n(\mathbb{R})$ be the group of invertible $n \times n$ matrices with entries from $\mathbb{R}$. Then $\mathrm{GL}_n(\mathbb{R})$ can be made into an involutive group by defining

$$\tau(A) = (A^{\mathsf{t}})^{-1}, \qquad A \in \mathrm{GL}_n(\mathbb{R}).$$

Here, A^{t} is the transpose of A. Clearly, $A^{\dagger} = A^{\mathsf{t}}$ for all $A \in \mathrm{GL}_n(\mathbb{R})$. In this case,

$$G(\mathrm{GL}_n(\mathbb{R})) = \{A \in \mathrm{GL}_n(\mathbb{R}) \colon A \text{ is symmetric and positive definite}\},$$
$$A(\mathrm{GL}_n(\mathbb{R})) = \{O \in \mathrm{GL}_n(\mathbb{R}) \colon O \text{ is orthogonal}\}.$$

Since $G(\mathrm{GL}_n(\mathbb{R}))$ is uniquely 2-divisible, it follows that $G(\mathrm{GL}_n(\mathbb{R}))$ is a gyrogroup under the operation

$$A \oplus B = \sqrt{AB^2A}$$

and any gyroautomorphism is a congruence transformation, $A \mapsto O^{\mathsf{t}}AO$, where O is an orthogonal matrix.

Concrete example—unital C^*-algebra

Positive units in a unital C^*-algebra

The set of positive units in a unital C^*-algebra is a gyrocommutative gyrogroup under the operation

$$x \oplus y = \sqrt{xy^2x}$$

and under the operation

$$x \oplus_H y = \sqrt{x}y\sqrt{x}.$$

In both cases, any gyroautomorphism is a congruence transformation, $x \mapsto uxu^*$, where u is a unitary element.

Commutator-inversion invariant groups

Recall that the *commutator* of g and h in a group Γ is denoted by $[g,h]$ and is defined as $[g,h] = g^{-1}h^{-1}gh$. Denote by $Z(\Gamma)$ the center of Γ given by

$$Z(\Gamma) = \{z \in \Gamma \colon zg = gz \text{ for all } g \in \Gamma\}.$$

Definition 7

A group Γ is commutator-inversion invariant if $[g,h] = [g^{-1},h^{-1}]$ for all $g,h \in \Gamma$ and is central by a commutator-inversion invariant group if $\Gamma/Z(\Gamma)$ is commutator-inversion invariant.

Construction of a gyrogroup II

Theorem 8 (T. S., 2022)

Let Γ be a group. If $\Gamma/Z(\Gamma)$ is commutator-inversion invariant, then Γ can be made into a gyrogroup by defining

$$a \oplus b = aaba^{-1} \tag{7}$$

for all $a, b \in \Gamma$. In this case, the induced gyrogroup is denoted by Γ^{gyr}. The gyroautomorphism of Γ^{gyr} generated by a and b is conjugation by $[a^{-1}, b]$.

Characterization when a gyroautomorphism is trivial

Recall that a group Γ is said to be *nilpotent* if its upper central series reaches Γ at some step.

Theorem 9

Let Γ be a group central by a commutator-inversion invariant group. Then every gyroautomorphism of Γ^{gyr} is trivial if and only if Γ is nilpotent of class at most 2.

Characterization when induced gyrogroups are isomorphic

A group Γ is said to be 3-divisible if for each element $g \in \Gamma$, there is an element $h \in \Gamma$ for which $g = h^3$.

Theorem 10

Let Γ and Π be groups central by commutator-inversion invariant groups. If Γ is 3-divisible, then Γ and Π are isomorphic as groups if and only if Γ^{gyr} and Π^{gyr} are isomorphic as gyrogroups.

Characterization when induced gyrogroups are isomorphic—finite case

Theorem 11

Let Γ and Π be finite groups central by commutator-inversion invariant groups. If $|\Gamma|$ is not divisible by 3, then Γ and Π are isomorphic as groups if and only if Γ^{gyr} and Π^{gyr} are isomorphic as gyrogroups.

Some examples of groups of prime-power order

Theorem 12

If Γ is a group of order p^k, where p is a prime and $k \in \{0,1,2,3\}$, then Γ^{gyr} exists and is degenerate.

Theorem 13

Let Γ be a group of order p^4, where p is a prime. Then Γ^{gyr} exists, and in this case Γ^{gyr} is degenerate if and only if Γ is nilpotent of class at most 2.

Concrete examples

The following groups are central by commutator-inversion invariant groups and produce non-degenerate gyrogroups:

1. the *dihedral group* of order 16, $D_{16} = \langle r, s : r^8 = s^2 = 1, rs = sr^{-1} \rangle$
2. the *generalized quaternion group* of order 16, $Q_{16} = \langle a, b : a^8 = 1, a^4 = b^2, bab^{-1} = a^{-1} \rangle$
3. the *semidihedral group* of order 16, $SD_{16} = \langle x, y : x^8 = y^2 = 1, yxy^{-1} = x^3 \rangle$

The three induced gyrogroups D_{16}^{gyr}, Q_{16}^{gyr}, and SD_{16}^{gyr} are pairwise non-isomorphic by Theorem 11.

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Thank you for your attention!