INTRODUCING UP-MODULES

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Conference on Recent Trends in Algebra and Related Topics 2023
Department of Mathematics, Faculty of Science, Chiang Mai University, Thailand
January 19-20, 2023

Topics

1. Abstract

The goal of this study is to introduce the concept of a new type of the hybrid algebra between Abelian groups and UP-algebras: UP-modules. We introduce the concept of fuzzy UP-submodules of UP-modules and provides properties and finds the necessary and sufficient conditions for this concept. We define fuzzy sets in UP-modules of many forms, supplying their properties and their relation to fuzzy UP-submodules. We also define and study the fuzzy UP-submodule generated by a set of fuzzy sets in UP-modules, as well as provide for their properties and their relation to fuzzy UP-submodules.

Finally, we apply the concept of fuzzy UP-ideals of UP-algebras while providing properties and find the results of the composition and the product between fuzzy UP-ideals and fuzzy UP-submodules.

2. Introduction

DEFINITION 2.1 [a] An algebra $X = (X; \cdot, 1)$ of type $(2, 0)$ is called a *UP-algebra*, where X is a nonempty set, $\cdot$ is a binary operation on X, and 1 is a fixed element of X if it satisfies the following axioms:

$$(\forall x, y, z \in X)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 1) \tag{UP-1}$$

$$(\forall x \in X)(1 \cdot x = x) \tag{UP-2}$$

$$(\forall x \in X)(x \cdot 1 = 1) \tag{UP-3}$$

$$(\forall x, y \in X)(x \cdot y = 1, y \cdot x = 1 \Rightarrow x = y) \tag{UP-4}$$

[a] A. Iampan. "A new branch of the logical algebra: UP-algebras". In: *J. Algebra Relat. Top.* 5.1 (2017), pp. 35–54. DOI: 10.22124/JART.2017.2403.

A partial ordering $\leq$ is defined on a UP-algebra $X = (X; \cdot, 1)$ by

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x \cdot y = 1). \tag{2.1}$$

In a UP-algebra $X = (X; \cdot, 1)$, the following assertions are valid (see[a],[b]).

$$(\forall x \in X)(x \cdot x = 1) \tag{2.2}$$

$$(\forall x, y, z \in X)(x \cdot y = 1, y \cdot z = 1 \Rightarrow x \cdot z = 1) \tag{2.3}$$

$$(\forall x, y, z \in X)(x \cdot y = 1 \Rightarrow (z \cdot x) \cdot (z \cdot y) = 1) \tag{2.4}$$

$$(\forall x, y, z \in X)(x \cdot y = 1 \Rightarrow (y \cdot z) \cdot (x \cdot z) = 1) \tag{2.5}$$

$$(\forall x, y \in X)(x \cdot (y \cdot x) = 1) \tag{2.6}$$

[a] Iampan, “A new branch of the logical algebra: UP-algebras”.

[b] A. Iampan. “Introducing fully UP-semigroups”. In: *Discuss. Math., Gen. Algebra Appl.* 38.2 (2018), pp. 297–306. DOI: 10.7151/dmgaa.1290.

$$(\forall x, y \in X)((y \cdot x) \cdot x = 1 \Leftrightarrow x = y \cdot x) \tag{2.7}$$

$$(\forall x, y \in X)(x \cdot (y \cdot y) = 1) \tag{2.8}$$

$$(\forall a, x, y, z \in X)((x \cdot (y \cdot z)) \cdot (x \cdot ((a \cdot y) \cdot (a \cdot z))) = 1) \tag{2.9}$$

$$(\forall a, x, y, z \in X)((((a \cdot x) \cdot (a \cdot y)) \cdot z) \cdot ((x \cdot y) \cdot z) = 1) \tag{2.10}$$

$$(\forall x, y, z \in X)(((x \cdot y) \cdot z) \cdot (y \cdot z) = 1) \tag{2.11}$$

$$(\forall x, y, z \in X)(x \cdot y = 1 \Rightarrow x \cdot (z \cdot y) = 1) \tag{2.12}$$

$$(\forall x, y, z \in X)(((x \cdot y) \cdot z) \cdot (x \cdot (y \cdot z)) = 1) \tag{2.13}$$

$$(\forall a, x, y, z \in X)(((x \cdot y) \cdot z) \cdot (y \cdot (a \cdot z)) = 1) \tag{2.14}$$

EXAMPLE 2.2 [a] Let U be a nonempty set and let $X \in \mathcal{P}(U)$ where $\mathcal{P}(U)$ means the power set of U. Let $\mathcal{P}_X(U) = \{A \in \mathcal{P}(U) \mid X \subseteq A\}$. Define a binary operation $\cdot$ on $\mathcal{P}_X(U)$ by putting $A \cdot B = B \cap (A' \cup X)$ for all $A, B \in \mathcal{P}_X(U)$ where A' means the complement of a subset A. Then $(\mathcal{P}_X(U), \cdot, X)$ is a UP-algebra. Let $\mathcal{P}^X(U) = \{A \in \mathcal{P}(U) \mid A \subseteq X\}$. Define a binary operation $*$ on $\mathcal{P}^X(U)$ by putting $A * B = B \cup (A' \cap X)$ for all $A, B \in \mathcal{P}^X(U)$. Then $(\mathcal{P}^X(U), *, X)$ is a UP-algebra.

[a] A. Satirad, P. Mosrijai, and A. Iampan. "Generalized power UP-algebras". In: *Int. J. Math. Comput. Sci.* 14.1 (2019), pp. 17–25.

DEFINITION 2.3 A UP-algebra $X = (X; \cdot, 1)$ is said to be

(i) *bounded* if there is an element $0 \in X$ such that $0 \leq x$ for all $x \in X$, that is,

$$(\forall x \in X)(0 \cdot x = 1), \tag{Bounded}$$

(ii) *meet-commutative* if it satisfies the identity

$$(\forall x, y \in X)(x \wedge y = y \wedge x), \tag{Meet-commutative}$$

where

$$(\forall x, y \in X)(x \wedge y = (y \cdot x) \cdot x). \tag{Meet}$$

3. Introducing UP-modules

In this section, we introduce a system of hybrid algebra between UP-algebras and Abelian groups in a form similar to the well-known modules. This new algebraic system is called UP-modules, which is defined as follows.

DEFINITION 3.1 By a *left UP-module* (briefly, *UP-module*) over a UP-algebra $X = (X; \cdot, 1)$, we mean an Abelian group $M = (M; +, 0)$ with an operation $X \times M \to M$ with $(x, m) \mapsto xm$ satisfies the following axioms:

$$(\forall x, y \in X, \forall m \in M)((x \wedge y)m = x(ym)) \quad \text{(UPM-1)}$$

$$(\forall x \in X, \forall m, n \in M)(x(m + n) = xm + xn) \quad \text{(UPM-2)}$$

$$(\forall m \in M)(1m = 0) \quad \text{(UPM-3)}$$

EXAMPLE 3.2 Let A be a nonempty set and $X = \mathcal{P}(A)$. Then $(X;+,\emptyset)$ is an Abelian group with $m+n = (m-n)\cup(n-m)$ for any $m,n \in X$. By Example 2.2, we get $(X;\cdot,\emptyset)$ is a UP-algebra. Hence, X is a UP-module over itself with $xm = x\cap m$ for all $x,m \in X$.

EXAMPLE 3.3 Let A be a nonempty set and $X = \mathcal{P}(A)$. Then $(X;+,A)$ is an Abelian group with $m+n=(m\cap n)\cup(n\cup m)'$ for any $m,n\in X$. By Example 2.2, we get $(X;*,A)$ is a UP-algebra. Hence, X is a UP-module over itself with $xm=x\cup m$ for all $x,m\in X$.

DEFINITION 3.4 A UP-module M over X is said to be

(i) *unitary* (when X is bounded) if it satisfies the identity

$$(\forall m \in M)(0m = m), \qquad \text{(Unitary)}$$

(ii) *separability* if it satisfies the identity

$$(\forall x \in X, \forall m \in M)(xm = m), \qquad \text{(Separability)}$$

(iii) *distributive* if it satisfies the identity

$$(\forall x, y \in X, \forall m, n \in M)(xm + yn = (x \wedge y)(m + n)). \qquad \text{(Distributive)}$$

For convenience, we define M as a UP-module M over X until further described, where we shall let $X = (X; \cdot, 1)$ be a UP-algebra and $M = (M; +, 0)$ an Abelian group.

PROPOSITION 3.5 *Let* $x, x_i \in X$ *and* $m, m_i \in M$ *for all* $i \in \{1, 2, \ldots, k\}$. *Then the following properties hold.*

(i) $(\forall x \in X, \forall m \in M)((1 \wedge x)m = 0)$,

(ii) $(\forall x \in X)(x0 = 0)$,

(iii) $(\forall x \in X, \forall m \in M)((x \wedge 1)m = 0)$,

(iv) $(\forall x \in X, \forall m \in M)((x \wedge x)m = xm)$,

(v) $(\forall x \in X, \forall m \in M)(-xm = x(-m))$,

(vi) $(\forall x \in X, \forall m, n \in M)(x(m - n) = xm - xn)$,

(vii) $-(\sum_{i=1}^{k} x_i m_i) = \sum_{i=1}^{k} x_i(-m_i)$.

DEFINITION 3.6 Let N be a subgroup of M. Then N is called a *UP-submodule* of M if N is a UP-module over X under the same multiplication which is defined on X and M.

THEOREM 3.7 *A nonempty subset A of M is a UP-submodule if and only if $a - b, xa \in A$ for all $x \in X$ and $a, b \in A$.*

4. Fuzzy sets in UP-modules

A *fuzzy set*[a] in a nonempty set X is defined to be a function $\mu : X \to [0,1]$, where $[0,1]$ is the unit closed interval of real numbers. We say that a fuzzy set in X is *constant* if it is a constant function. We define 0_X and 1_X represent the constant fuzzy sets in X that map every element of X to 0 and every element of X to 1, respectively.

[a] L. A. Zadeh. “Fuzzy Sets”. In: *Inf. Cont.* 8.3 (1965), pp. 338–353.

DEFINITION 4.1 A fuzzy set α in M is called a *fuzzy UP-submodule* of M if the following axioms hold:

$$(\forall m, n \in M)(\alpha(m+n) \geq \min\{\alpha(m), \alpha(n)\}) \qquad \text{(FUPSM-1)}$$

$$(\forall m \in M)(\alpha(-m) = \alpha(m)) \qquad \text{(FUPSM-2)}$$

$$(\forall x \in X, \forall m \in M)(\alpha(xm) \geq \alpha(m)) \qquad \text{(FUPSM-3)}$$

From now on, we define $F(M), FS(M)$, and $F(X)$ as the set of all fuzzy sets and fuzzy UP-submodules of a UP-module M over X, and the set of all fuzzy sets in X, respectively.

The binary relation $\leq$ on $F(M)$ is defined as follows:

$$(\forall \alpha, \beta \in F(M))(\alpha \leq \beta \Leftrightarrow (\forall m \in M)(\alpha(m) \leq \beta(m))).$$

The binary relation $\leq$ on $F(X)$ is defined the same as on $F(M)$.

EXAMPLE 4.2 Let $X = \{0, 1, 2, 3\}$ be a set with two binary operations $\cdot$ and $+$ defined by the following tables:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	1
3	0	0	0	0

$+$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then $X = (X; \cdot, 0)$ is a UP-algebra and $X = (X; +, 0)$ is an Abelian group.

Thus X is a UP-module over itself with an operation defined by the following table:

	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

Now, let $t_0, t_1 \in [0,1]$ be such that $t_0 < t_1$. We define a fuzzy set α on X as follows:

$$\alpha = \begin{pmatrix} 0 & 1 & 2 & 3 \\ t_1 & t_0 & t_0 & t_0 \end{pmatrix}.$$

Hence, α is a fuzzy UP-submodule of X.

THEOREM 4.3 *If* $\alpha \in F(M)$ *satisfies* (FUPSM-3), *then*

$$(\forall m \in M)(\alpha(0) \geq \alpha(m)). \tag{4.1}$$

THEOREM 4.4 *Let* $\alpha \in F(M)$. *Then* $\alpha \in FS(M)$ *if and only if it satisfies* (FUPSM-3) *and*

$$(\forall m, n \in M)(\alpha(m - n) \geq \min\{\alpha(m), \alpha(n)\}). \tag{4.2}$$

THEOREM 4.5 *Let* M *be unitary and* $\alpha \in F(M)$. *Then* $\alpha \in FS(M)$ *if and only if it satisfies* (4.1) *and*

$$(\forall x, y \in X, \forall m, n \in M)(\alpha(xm - yn) \geq \min\{\alpha(m), \alpha(n)\}). \tag{4.3}$$

DEFINITION 4.6 Let $\alpha \in F(M)$. For all $t \in [0,1]$, the set

$$U(\alpha; t) = \{m \in M \mid \alpha(m) \geq t\}$$

is called an *upper t-level subset* of α.

THEOREM 4.7 *Let $\alpha \in F(M)$. Then $\alpha \in FS(M)$ if and only if for all $t \in [0,1], \emptyset \neq U(\alpha; t)$ is a UP-submodule of M.*

5. Some properties of fuzzy sets in UP-modules

DEFINITION 5.1 Let $k \in \mathbb{N}, \alpha, \alpha_i \in F(M)$ for all $i \in \{1, 2, \ldots, k\}$, and $x \in X$. We define fuzzy sets $\sum_{i=1}^{k} \alpha_i, -\alpha$, and $x\alpha$ in M as follows:

$$(\forall m \in M)((\sum_{i=1}^{k} \alpha_i)(m) = \sup_{m=\sum_{i=1}^{k} a_i} \{\min_{i=1}^{k}\{\alpha_i(a_i)\}\})$$

$$(\forall m \in M)((-\alpha)(m) = \alpha(-m))$$

$$(\forall m \in M)((x\alpha)(m) = \sup_{m=xn} \{\alpha(n)\})$$

DEFINITION 5.2 For all $i \in \{1, 2, \ldots, k\}, \alpha_i \in F(M)$ is said to have the *same tip* if $\alpha_i(0) = \alpha_j(0)$ for all $i, j \in \{1, 2, \ldots, k\}$.

PROPOSITION 5.3 *Let* $\alpha_i, \alpha, \beta, \gamma \in F(M)$ *for all* $i \in \{1, 2, \ldots, k\}$. *Then the following statements hold:*

(i) $(1\alpha)(0) \geq \alpha(m)$ *for all* $m \in M$,

(ii) *if* M *is unitary, then* $0\alpha = \alpha$,

(iii) *if* $\alpha \leq \beta$, *then* $x\alpha \leq x\beta$ *for all* $x \in X$,

(iv) *if* M *is unitary and* $0\alpha \leq 0\beta$, *then* $x\alpha \leq x\beta$ *for all* $x \in X$,

(v) $(x \wedge y)\alpha = x(y\alpha)$ *for all* $x, y \in X$,

(vi) *if* $\alpha_i \leq \beta_i$ *for all* $i \in \{1, 2, \ldots, k\}$, *then* $\sum_{i=1}^{k} \alpha_i \leq \sum_{i=1}^{k} \beta_i$,

(vii) $\alpha(m) \leq (x\alpha)(xm)$ *for all* $x \in X$ *and* $m \in M$,

(viii) $(\forall x \in X)(\alpha(m) \leq \gamma(xm)$ *for all* $m \in M$ *if and only if* $x\alpha \leq \gamma)$,

(ix) *if* $\alpha_i \in FS(M)$ *and has the same tip for all* $i \in \{1, 2, \ldots, k\}$, *then* $\alpha_i \leq \sum_{i=1}^{k} \alpha_i$,

(x) $(x\alpha + y\beta)(xm + yn) \geq \min\{\alpha(m), \beta(n)\}$ *for all* $x, y \in X$ *and* $m, n \in M$,

(xi) $(x\alpha + y\beta)(xm - yn) \geq \min\{\alpha(m), \beta(-n)\}$ *for all* $x, y \in X$ *and* $m, n \in M$, *in particular, if* (FUPSM-2) *holds, then* $(x\alpha+y\beta)(xm-yn) \geq \min\{\alpha(m), \beta(n)\}$ *for all* $x, y \in X$ *and* $m, n \in M$,

(xii) *if* $\gamma \geq x\alpha + y\beta$ *for* $x, y \in X$, *then* $\gamma(xm - yn) \geq \min\{\alpha(m), \beta(-n)\}$ *for all* $m, n \in M$, *in particular, if* (FUPSM-2) *holds, then* $\gamma(xm - yn) \geq \min\{\alpha(m), \beta(n)\}$ *for all* $m, n \in M$.

THEOREM 5.4 *If* $\alpha \in FS(M)$, *then it satisfies* (FUPSM-2) *and*

$$(\forall x \in X)(x\alpha \leq \alpha), \tag{5.1}$$

$$\alpha + \alpha \leq \alpha. \tag{5.2}$$

COROLLARY 5.5 *If* $\alpha \in FS(M)$, *then*

$$\sum_{i=1}^{k} \alpha \leq \alpha. \tag{5.3}$$

THEOREM 5.6 *Let* M *be unitary and* $\alpha \in F(M)$. *If* α *satisfies* (FUPSM-2), (5.1), *and* (5.2), *then* $\alpha \in FS(M)$.

THEOREM 5.7 *Let M be unitary and $\alpha \in F(M)$. If α satisfies* (4.1), (FUPSM-2), *and*

$$(\forall x, y \in X)(x\alpha + y\alpha \leq \alpha), \tag{5.4}$$

then $\alpha \in FS(M)$.

THEOREM 5.8 *Let $\alpha \in FS(M)$. Then the following statements hold:*

(i) $-\alpha \in FS(M)$,

(ii) *if M is unitary, X is meet-commutative, and $x\alpha$ satisfies* (5.2) *for $x \in X$, that is, $x\alpha + x\alpha \leq x\alpha$, then $x\alpha \in FS(M)$.*

DEFINITION 5.9 Let $\{\alpha_i \mid i \in \Lambda\} \subseteq F(M)$. We define fuzzy sets $\bigcap_{i\in\Lambda} \alpha_i$ and $\bigcup_{i\in\Lambda} \alpha_i$ in M as follows:

$$(\forall m \in M)((\bigcap_{i\in\Lambda} \alpha_i)(m) = \inf_{i\in\Lambda}\{\alpha_i(m)\}),$$

$$(\forall m \in M)((\bigcup_{i\in\Lambda} \alpha_i)(m) = \sup_{i\in\Lambda}\{\alpha_i(m)\}).$$

LEMMA 5.10 *Let $\beta, \alpha_i \in F(M)$. Then the following statements hold:*

(i) *if $\beta \leq \alpha_i$ for all $i \in \Lambda$, then $\beta \leq \bigcap_{i\in\Lambda} \alpha_i$,*

(ii) *if $\alpha_i \leq \beta$ for all $i \in \Lambda$, then $\bigcup_{i\in\Lambda} \alpha_i \leq \beta$.*

THEOREM 5.11 $(F(M), \cup, \cap)$ *is a complete lattice.*

THEOREM 5.12 *If* $\alpha_i \in FS(M)$ *for all* $i \in \Lambda$*, then* $\bigcap_{i\in\Lambda} \alpha_i \in FS(M)$*.*

THEOREM 5.13 *If* $\alpha_i \in F(M)$ *satisfies* (FUPSM-2) *and* (FUPSM-3) *for all* $i \in \Lambda$*, then* $\bigcup_{i\in\Lambda} \alpha_i$ *satisfies* (FUPSM-2) *and* (FUPSM-3)*, respectively.*

DEFINITION 5.14 Let Y be a subset of a set X. The characteristic function of Y is defined as follows:

$$(\forall x \in X)\left(\chi_Y(x) = \begin{cases} 1 & \text{if } x \in Y \\ 0 & \text{otherwise} \end{cases}\right).$$

In particular, $\chi_\emptyset = 0_X$ and $\chi_X = 1_X$.

DEFINITION 5.15 Let a be an element of a set X and $t \in [0, 1]$. The fuzzy point a_t in X is defined as follows:

$$(\forall x \in X) \left(a_t(x) = \begin{cases} t & \text{if } x = a \\ 0 & \text{otherwise} \end{cases} \right).$$

6. Fuzzy UP-submodule generated by a set

In this section, we define and study the fuzzy UP-submodule generated by a set of fuzzy sets in UP-modules, as well as provide for their properties and their relation to fuzzy UP-submodules.

DEFINITION 6.1 Let $A \subseteq F(M)$. The intersection of all fuzzy UP-submodules of M greater than all fuzzy sets in A is called the *fuzzy UP-submodule generated by* A, denoted by $\langle A \rangle$. By Theorem 5.12, we get $\langle A \rangle$ is the least fuzzy UP-submodule of M greater than all fuzzy sets in A. If $A = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$, then we write $\langle A \rangle = \langle \alpha_1, \alpha_2, \ldots, \alpha_k \rangle$. If A is finite and $\alpha = \langle A \rangle$, then we say that α is finitely generated. In particular, if $\alpha = \langle \alpha_t \rangle$, then we say that α is *cyclic*.

DEFINITION 6.2 Let N be a subset of M. We define a subset $[N]$ of M as follows:

$$[N] = \{m \in M \mid m = xn \text{ for some } x \in X \text{ and } n \in N\}.$$

LEMMA 6.3 *Let N be a subset of M. Then*

(i) *if M is unitary, then $N \subseteq [N]$,*

(ii) *if N is a UP-submodule of M, then $[N] \subseteq N$,*

(iii) *if M is unitary and N is a UP-submodule of M, then $N = [N]$.*

LEMMA 6.4 *Let* $\alpha, \beta \in F(M)$. *Then*

(i) *if* α *satisfies* (FUPSM-1), *then* $U(\alpha; s) + U(\alpha; t) \subseteq U(\alpha; \min\{s, t\})$ *for all* $s, t \in [0, 1]$,

(ii) *if* α *satisfies* (FUPSM-1) *and* (FUPSM-3), *then* $[U(\alpha; s)] + [U(\alpha; t)] \subseteq U(\alpha; \min\{s, t\})$ *for all* $s, t \in [0, 1]$,

(iii) *if* $\alpha \leq \beta$, *then* $U(\alpha; t) \subseteq U(\beta; t)$ *for all* $t \in [0, 1]$,

(iv) *if* $\alpha \leq \beta$ *and* β *satisfies* (FUPSM-3), *then* $[U(\alpha; t)] \subseteq U(\beta; t)$ *for all* $t \in [0, 1]$.

COROLLARY 6.5 *Let M be unitary and $\alpha, \beta \in F(M)$. Then*

(i) *if α satisfies* (FUPSM-1), *then* $U(\alpha; s) + U(\alpha; t) \subseteq U(\alpha; \min\{s, t\}) \subseteq [U(\alpha; \min\{s, t\})]$ *for all* $s, t \in [0, 1]$,

(ii) *if α satisfies* (FUPSM-1) *and* (FUPSM-3), *then* $U(\alpha; s) + U(\alpha; t) \subseteq [U(\alpha; s)] + [U(\alpha; t)] \subseteq U(\alpha; \min\{s, t\}) \subseteq [U(\alpha; \min\{s, t\})]$ *for all* $s, t \in [0, 1]$,

(iii) *if* $\alpha \leq \beta$, *then* $U(\alpha; t) \subseteq U(\beta; t) \subseteq [U(\beta; t)]$ *for all* $t \in [0, 1]$,

(iv) *if* $\alpha \leq \beta$ *and β satisfies* (FUPSM-3), *then* $U(\alpha; t) \subseteq [U(\alpha; t)] \subseteq U(\beta; t) \subseteq [U(\beta; t)]$ *for all* $t \in [0, 1]$.

LEMMA 6.6 *If* $\alpha \in F(M)$ *satisfies* (FUPSM-3), *then*

$$(\forall x \in X)(\sup\{t \in [0,1] \mid xm \in U(\alpha;t)\} \geq \sup\{t \in [0,1] \mid m \in U(\alpha;t)\}). \tag{6.1}$$

DEFINITION 6.7 Let $f \in F(X)$ and $\beta \in F(M)$. The composition $f \circ \beta$ and the product $f\beta$ of f and β are defined as follows:

$$(\forall m \in M)((f \circ \beta)(m) = \sup_{m=xn} \{\min\{f(x), \beta(n)\}\}),$$
$$(\forall m \in M)((f\beta)(m) = \sup_{m=\sum_{i=1}^{k} x_i m_i} \{\min\{f(x_1), \ldots, f(x_k), \beta(m_1), \ldots, \beta(m_k)\}$$

From Definition 6.7, we know that

$$(\forall f \in F(X), \forall \beta \in F(M))(f \circ \beta \leq f\beta). \tag{6.2}$$

THEOREM 6.8 *Let* $A = \{\alpha_i \mid i \in I\} \subseteq FS(M)$. *Then*

(i) *if* $\bigcup_{i\in I} \alpha_i$ *satisfies* (FUPSM-1), *then*

$$(\forall m \in M)(\langle A\rangle(m) = \sup\{t \in [0,1] \mid m \in U(\bigcup_{i\in I} \alpha_i; t)\}),$$

(ii) *if* M *is separability, then* $a_t = 1_X \circ a_t$ *for all* $a \in M$ *and* $t \in [0,1]$,

(iii) $\langle 0_t \rangle = 0_t$ *for all* $t \in [0,1]$,

(iv) *if* M *is separability, then* $\bigcup_{a_t \le \alpha}(1_X \circ a_t) \le \alpha$ *for all* $\alpha \in F(M)$.

THEOREM 6.9 *Let* $A = \{\alpha_i \mid i \in \{1, 2, \ldots, k\}\} \subseteq FS(M)$ *with the same tip. Then* $\bigcup_{i=1}^{k} \alpha_i \leq \sum_{i=1}^{k} \alpha_i$. *Moreover, if* $\sum_{i=1}^{k} \alpha_i$ *is a fuzzy UP-submodule of* M*, then* $\langle \bigcup_{i=1}^{k} \alpha_i \rangle = \sum_{i=1}^{k} \alpha_i$.

THEOREM 6.10 *Let* $\alpha, \beta, \gamma \in F(M)$. *If* α *satisfies* (FUPSM-1), *then*

$$\alpha \cap (\beta + \gamma) \geq (\alpha \cap \beta) + (\alpha \cap \gamma).$$

7. Fuzzy UP-ideals of UP-algebras

DEFINITION 7.1 [a] A fuzzy set f in X is called a *fuzzy UP-ideal* of X if it satisfies the following properties:

$$(\forall x \in X)(f(1) \geq f(x)), \tag{7.1}$$

$$(\forall x, y, z \in X)(f(x \cdot z) \geq \min\{f(x \cdot (y \cdot z)), f(y)\}). \tag{7.2}$$

We define $FI(X)$ as the set of all fuzzy UP-ideals of X.

[a] J. Somjanta et al. “Fuzzy Sets in UP-algebras”. In: *Ann. Fuzzy Math. Inform.* 12.6 (2016), pp. 739–756.

PROPOSITION 7.2 *If* $f \in FI(X)$, *then*

$$(\forall x, y \in X)(f(x \wedge y) \geq \max\{f(x), f(y)\}). \tag{7.3}$$

DEFINITION 7.3 An $\alpha \in F(M)$ is said to be *increasing* if it satisfies the identity

$$(\forall m, n \in M)(\alpha(m+n) \geq \max\{\alpha(m), \alpha(n)\}). \quad \text{(Increasing)}$$

We know that every increasing fuzzy set in a UP-module satisfies (FUPSM-1).

LEMMA 7.4 *If $\alpha \in F(M)$ satisfies (FUPSM-3) and is increasing, then*

$$(\forall m \in M)(\alpha(m) = \alpha(0)). \tag{7.4}$$

THEOREM 7.5 *If M is distributive, $f \in FI(X)$, and $\beta \in F(M)$ is increasing, then $f \circ \beta$ is increasing, that is, it satisfies (FUPSM-1).*

THEOREM 7.6 *If $f \in FI(X)$ and $\beta \in F(M)$ satisfies (FUPSM-2), then $f \circ \beta$ satisfies (FUPSM-2) and (FUPSM-3).*

THEOREM 7.7 *If M is distributive, $f \in FI(X)$, and $\beta \in F(M)$ is increasing and satisfies (FUPSM-2), then $f \circ \beta \in FS(M)$.*

THEOREM 7.8 *If* M *is distributive,* $f \in FI(X)$*, and* $\beta \in F(M)$ *is increasing, then* $f\beta$ *is increasing, that is, it satisfies* (FUPSM-1).

THEOREM 7.9 *If* $f \in FI(X)$ *and* $\beta \in F(M)$ *satisfies* (FUPSM-2), *then* $f\beta$ *satisfies* (FUPSM-2) *and* (FUPSM-3).

THEOREM 7.10 *If* M *is distributive,* $f \in FI(X)$*, and* $\beta \in F(M)$ *is increasing and satisfies* (FUPSM-2), *then* $f\beta \in FS(M)$.

PROPOSITION 7.11 *Let* $f, g \in F(X)$ *and* $\alpha, \beta \in F(M)$, *where* β *satisfies* (FUPSM-1). *Then* $f \circ \alpha \leq \beta$ *if and only if* $f\alpha \leq \beta$.

PROPOSITION 7.12 *Let* $f, g \in F(X)$ *and* $\alpha, \beta \in F(M)$. *Then*

(i) *if* $\alpha \leq \beta$, *then* $f \circ \alpha \leq f \circ \beta$ *and* $f\alpha \leq f\beta$,

(ii) *if* $f \leq g$, *then* $f \circ \beta \leq g \circ \beta$ *and* $f\beta \leq g\beta$.

LEMMA 7.13 *Let* $x_s, x_t, f \in F(X)$ *and* $a_s, a_t, \beta \in F(M)$. *Then*

(i) $f \circ a_{\min\{t,s\}} \leq (f \circ a_s) \cap (f \circ a_t)$,

(ii) $fa_{\min\{t,s\}} \leq (fa_s) \cap (fa_t)$,

(iii) $x_{\min\{t,s\}} \circ \beta \leq (x_s \circ \beta) \cap (x_t \circ \beta)$,

(iv) $x_{\min\{t,s\}}\beta \leq (x_s\beta) \cap (x_t\beta)$.

LEMMA 7.14 *Let* $f \in F(X)$ *and* $a_t, b_s, \beta \in F(M)$, *where* β *satisfies* (FUPSM-1). *If* $f \circ a_t \leq \beta$ *and* $f \circ b_s \leq \beta$, *then*

$$(\forall x \in X)((f \circ (a+b)_{\min\{t,s\}})(x(a+b)) \leq \beta(x(a+b))). \tag{7.5}$$

LEMMA 7.15 *Let $f \in F(X)$ and $a_t, b_s, \beta \in F(M)$, where β satisfies* (FUPSM-1) *and* (FUPSM-2). *If $f \circ a_t \leq \beta$ and $f \circ b_s \leq \beta$, then*

$$(\forall x \in X)((f \circ (a-b)_{\min\{t,s\}})(x(a-b)) \leq \beta(x(a-b))). \tag{7.6}$$

PROPOSITION 7.16 *Let $f \in FI(X)$ and $a_t \in F(M)$. Then*

$$(\forall x, y \in X)(f \circ (xa)_t)(y(xa)) \leq (f \circ a_t)(y(xa)). \tag{7.7}$$

LEMMA 7.17 *Let $x_t \in F(X)$ and $\alpha \in F(M)$. If α satisfies* (FUPSM-3), *then*

$$(1_t \circ \alpha)(0) \leq (x_t \circ \alpha)(0). \tag{7.8}$$

LEMMA 7.18 *Let $f \in F(X)$ and $a_t \in F(M)$. If f satisfies (7.1), then*

$$(f \circ 0_t)(0) \leq (f \circ a_t)(0). \tag{7.9}$$

LEMMA 7.19 *Let $f \in F(X)$ and $a_t, b_s, \beta \in F(M)$, where β satisfies (FUPSM-1). If $fa_t \leq \beta$ and $fb_s \leq \beta$, then*

$$(\forall x \in X)((f(a+b)_{\min\{t,s\}})(x(a+b)) \leq \beta(x(a+b))). \tag{7.10}$$

LEMMA 7.20 *Let $f \in F(X)$ and $a_t, b_s, \beta \in F(M)$, where β satisfies (FUPSM-1) and (FUPSM-2). If $fa_t \leq \beta$ and $fb_s \leq \beta$, then*

$$(\forall x \in X)((f(a-b)_{\min\{t,s\}})(x(a-b)) \leq \beta(x(a-b))). \tag{7.11}$$

PROPOSITION 7.21 *Let* $f \in FI(X)$ *and* $a_t \in F(M)$. *Then*

$$(\forall x, y \in X)(f(xa)_t)(y(xa)) \leq (fa_t)(y(xa)). \tag{7.12}$$

THANK YOU

for your time and attention