Department of Mathematics
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Semirings and *k*-ideals

by

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Semirings

Semirings (First notion in 1934)

H. S. Vandiver, *Note on a simple type of algebra in which the cancellation law of addition does not hold*, Bulletin of the American Mathematical Society, 40, 1934, 914-920.

NOTE ON A SIMPLE TYPE OF ALGEBRA IN WHICH THE CANCELLATION LAW OF ADDITION DOES NOT HOLD

BY H. S. VANDIVER

1. *Introduction.* I do not imagine that the algebraic system considered in this note can be new, but if it has been overlooked this has probably happened because of its simplicity. However, we shall be most interested here in examining the connection of the system with the foundations of ordinary algebra. As we shall see, the symbols employed have most of the properties of rational integers, the principal exceptions being that they are finite in number and from

$$a + b = a + c$$

we cannot infer $b = c$ in general.*

2. *Description of the System.* Suppose we introduce the natural numbers 1, 2, 3, · · · , employing for their use Peano's system

* In a system in which we may always infer $b = c$ under the condition given we shall say the cancellation law holds.

A *semiring* is an algebraic structure $(S, +, \cdot)$ such that $(S, +)$ and $(S, \cdot)$ are semigroups and

$$a \cdot (b + c) = a \cdot b + a \cdot c,$$
$$(a + b) \cdot c = a \cdot c + b \cdot c$$

for all $a, b, c \in S$.

Example 1. $(\mathbb{N}, +, \cdot)$ **and** $(\mathbb{N}, \max, \min)$ **are semirings.**

2. The structure $(S, +, \cdot)$ such that $(S, +)$ and $(S, \cdot)$ are left zero and right zero semigroups, respectively is a semiring.

Introduction

Let $(S,+,.)$ be a semiring.

- A semiring $(S,+,.)$ is called additively commutative if $(S,+)$ is commutative.
- An element $0 \in S$ is called an additive zero if $0 + x = x = x + 0$ for all $x \in S$.
- An element $0 \in S$ is called a multiplicative zero if $0x = 0 = x0$ for all $x \in S$.
- If $0 \in S$ is both an additive zero and a multiplicative zero the it is called an absorbing zero (or a zero element).

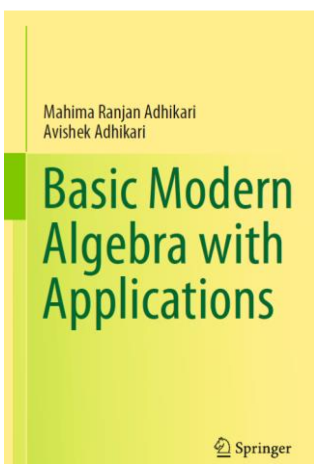
Introduction

Note: additive zero and multiplicative zero may not coincide.

Example: [M. R . & A. Adhikari; 2014] Consider the semiring $(\mathbb{N}_0, +, \cdot)$ where $\mathbb{N}_0$ is the set of all nonnegative integers, $\cdot$ is the usual multiplication and + is defined by

$$a + b = \begin{cases} \mathrm{lcm}(a, b)\,, & a \neq 0 \text{ and } b \neq 0 \\ 0\,, & \text{otherwise} \end{cases}$$

Now, the additive zero is 1 and the multiplicative zero is 0.



Semirings and weighted automata

Finite Automata (FA): $\mathcal{A} = (Q, T, I, F)$

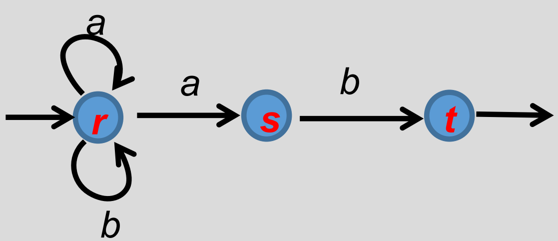
Q – finite set of states
$T \subseteq Q \times A \times Q$ - set of transitions
$I, F \subseteq Q$ – sets of initial resp. final states

A – an alphabet (a set of letters)
$w = a_1 \cdots a_n \in A^*$ is *accepted/recognized* by $\mathcal{A} \Leftrightarrow$
$\exists t_1, \cdots, t_n \in T, t_i = (q_{i-1}, a_1, q_i), q_0 \in I$ and $q_n \in F$

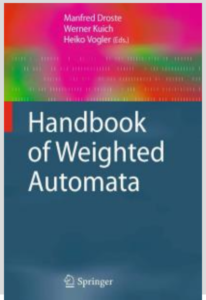
$$L(\mathcal{A}) = \{w \in A^* \mid \mathcal{A} \text{ accepts } w\}$$

e.g. $\mathcal{A}$

$A = \{a, b\}$
$Q = \{r, s, t\}$
$I = \{r\}$
$F = \{t\}$



$$L(\mathcal{A}) = \{w \in A^* \mid w \text{ ends with } ab\}$$



Weighted Finite Automata (WFA): $\mathcal{A} = (Q, \mathrm{wt}, \mathrm{in}, \mathrm{out})$

S – semiring, A – alphabet

Q – finite set of states

$\mathrm{wt}: Q \times A \times Q \to S$ – weight function

$\mathrm{in}, \mathrm{out}: Q \to S$ determine the weight/cost for entering resp., leaving $\mathcal{A}$ in state q

Path: $P = q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} q_{n-1} \xrightarrow{a_n} q_n$ (t_1, t_2, …, t_n)

$$\mathrm{weight}(P) = \mathrm{in}(q_0) \cdot \mathrm{wt}(t_1) \cdot \ldots \cdot \mathrm{wt}(t_n) \cdot \mathrm{out}(q_n)$$
where $t_i = (q_{i-1}, a_i, q_i)$

$\| \mathcal{A} \|: A^* \to S$ behavior of $\mathcal{A}$

$$\| \mathcal{A} \| (w) = \sum_{P \text{ path for } w} \mathrm{weight}(P)$$

M. Droste, W. Kuich, H. Vogler (eds.), *Handbook of Weighted Automata*, Monographs in Theoretical Computer Science. An EATCS Series, Springer-Verlag Berlin Heidelberg 2009

Semirings and weighted automata

Example: Finite Automata: $\mathcal{A} = (Q, T, I, F)$ **over alphabet** A

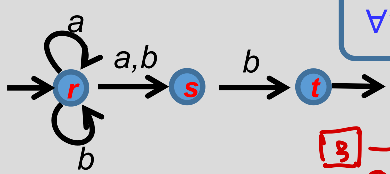
1) Let $S = (B, \vee, \wedge)$ with $B = \{0,1\}$ be the Boolean semiring.
Define a WFA $\mathcal{A}' = (Q, \mathrm{wt}, \mathrm{in}, \mathrm{out})$ as follows:

$$\mathrm{wt}(p,a,q) = \begin{cases} 1, & (p,a,q) \in T \\ 0, & \text{otherwise} \end{cases}, \quad \mathrm{in}(q) = \begin{cases} 1, & q \in I \\ 0, & q \notin I \end{cases}, \quad \text{and} \quad \mathrm{out}(q) = \begin{cases} 1, & q \in F \\ 0, & q \notin F \end{cases}$$

Then $\mathcal{A}'$ **is** a WFA over A and S and

$$\forall w \in A^*: \ \| \mathcal{A}' \| (w) = 1 \Leftrightarrow w \in L(\mathcal{A})$$

e.g. FA: $\mathcal{A}$



3 –a→ 3 –b→ 3 –a→ 3
s –a→ t ⇔ (s,a,b) ∉ T
P1 : r –a→ r –b→ r –a→ r
P2 r –a→ r –b→ r –a→ s

$$\| \mathcal{A}' \| (aba) = \big(\mathrm{in}(r) \wedge \mathrm{wt}((r,a,r)) \wedge \mathrm{wt}((r,b,r)) \wedge \mathrm{wt}((r,a,r)) \wedge \mathrm{out}(r)\big) \vee$$
$$\big(\mathrm{in}(r) \wedge \mathrm{wt}((r,a,r)) \wedge \mathrm{wt}((r,b,r)) \wedge \mathrm{wt}((r,a,s)) \wedge \mathrm{out}(s)\big)$$
$$= (1 \wedge 1 \wedge 1 \wedge 1 \wedge 0) \vee (1 \wedge 1 \wedge 1 \wedge 1 \wedge 0) = 0$$

$$\| \mathcal{A}' \| (abb) = \big(\mathrm{in}(r) \wedge \mathrm{wt}((r,a,r)) \wedge \mathrm{wt}((r,b,r)) \wedge \mathrm{wt}((r,b,r)) \wedge \mathrm{out}(r)\big) \vee$$
$$\big(\mathrm{in}(r) \wedge \mathrm{wt}((r,a,r)) \wedge \mathrm{wt}((r,b,r)) \wedge \mathrm{wt}((r,b,s)) \wedge \mathrm{out}(s)\big) \vee$$
$$\big(\mathrm{in}(r) \wedge \mathrm{wt}((r,a,r)) \wedge \mathrm{wt}((r,b,s)) \wedge \mathrm{wt}((s,b,t)) \wedge \mathrm{out}(t)\big)$$
$$= (1 \wedge 1 \wedge 1 \wedge 1 \wedge 0) \vee (1 \wedge 1 \wedge 1 \wedge 1 \wedge 0) \vee (1 \wedge 1 \wedge 1 \wedge 1 \wedge 1) = 1$$

Semirings and weighted automata

Example: Finite Automata: $\mathcal{A} = (\boldsymbol{Q}, \boldsymbol{T}, \boldsymbol{I}, \boldsymbol{F})$ **over alphabet *A***

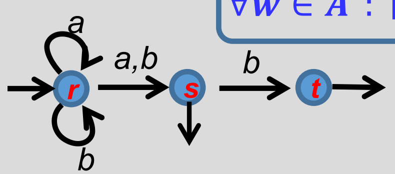
1) Let $S = (\mathbb{N}_0, +, \cdot)$ the semiring of nonnegative integers.
Define a WFA $\mathcal{A}' = (Q, \mathrm{wt}, \mathrm{in}, \mathrm{out})$ as follows:

$$\mathrm{wt}(p,a,q) = \begin{cases} 1, & (p,a,q) \in T \\ 0, & \text{otherwise} \end{cases}, \quad \mathrm{in}(q) = \begin{cases} 1, & q \in I \\ 0, & q \notin I \end{cases}, \quad \text{and} \quad \mathrm{out}(q) = \begin{cases} 1, & q \in I \\ 0, & q \notin I \end{cases}$$

Then $\mathcal{A}'$ is a WFA over *A* and *S* and

$$\forall \boldsymbol{w} \in \boldsymbol{A}^*: \parallel \mathcal{A}' \parallel (\boldsymbol{w}) = \#(\textbf{successful runs for } \boldsymbol{w} \textbf{ in } \mathcal{A})$$

e.g. FA: $\mathcal{A}$



$$\begin{aligned} \parallel \mathcal{A}' \parallel (\boldsymbol{aba}) &= \big(\mathrm{in}(r) \cdot \mathrm{wt}((r,a,r)) \cdot \mathrm{wt}((r,b,r)) \cdot \mathrm{wt}((r,a,r)) \cdot \mathrm{out}(r)\big) + \\ &\quad \big(\mathrm{in}(r) \cdot \mathrm{wt}((r,a,r)) \cdot \mathrm{wt}((r,b,r)) \cdot \mathrm{wt}((r,a,s)) \cdot \mathrm{out}(s)\big) \\ &= (1 \cdot 1 \cdot 1 \cdot 1 \cdot 0) + (1 \cdot 1 \cdot 1 \cdot 1 \cdot 1) = 1 \end{aligned}$$

$$\begin{aligned} \parallel \mathcal{A}' \parallel (\boldsymbol{abb}) &= \big(\mathrm{in}(r) \cdot \mathrm{wt}((r,a,r)) \cdot \mathrm{wt}((r,b,r)) \cdot \mathrm{wt}((r,b,r)) \cdot \mathrm{out}(r)\big) + \\ &\quad \big(\mathrm{in}(r) \cdot \mathrm{wt}((r,a,r)) \cdot \mathrm{wt}((r,b,r)) \cdot \mathrm{wt}((r,b,s)) \cdot \mathrm{out}(s)\big) + \\ &\quad \big(\mathrm{in}(r) \cdot \mathrm{wt}((r,a,r)) \cdot \mathrm{wt}((r,b,s)) \cdot \mathrm{wt}((s,b,t)) \cdot \mathrm{out}(t)\big) \\ &= (1 \cdot 1 \cdot 1 \cdot 1 \cdot 0) + (1 \cdot 1 \cdot 1 \cdot 1 \cdot 1) + (1 \cdot 1 \cdot 1 \cdot 1 \cdot 1) = 2 \end{aligned}$$

Many kinds of ideals of semirings

Let $(S,+,.)$ be a semiring and $\emptyset \neq A \subseteq S$.

ideals	definitions
Left ideal	$A + A \subseteq A$ and $SA \subseteq A$
Right ideal	$A + A \subseteq A$ and $AS \subseteq A$
Ideal	$A + A \subseteq A, SA \subseteq A$ and $AS \subseteq A$
Quasi-ideal	$A + A \subseteq A$ and $SA \cap AS \subseteq A$
Bi-ideal	$A + A \subseteq A, AA \subseteq A$ and $ASA \subseteq A$
Interior ideal	$A + A \subseteq A, AA \subseteq A$ and $SAS \subseteq A$

Many kinds of *k*-ideals of semirings

Let $(S,+,.)$ be an additively commutative semiring and $\emptyset \neq A \subseteq S$.

$$\bar{A} = \{x \in S \mid x + a \in A \text{ for some } a \in A\}$$

k-ideals	definitions
k-left ideal	$A + A \subseteq A$ and $SA \subseteq A$
k-right ideal	$A + A \subseteq A$ and $AS \subseteq A$
k-ideal	$A + A \subseteq A, SA \subseteq A$ and $AS \subseteq A$
k-quasi-ideal	$A + A \subseteq A$ and $SA \cap AS \subseteq A$
k-bi-ideal	$A + A \subseteq A, AA \subseteq A$ and $ASA \subseteq A$
k-interior ideal	$A + A \subseteq A, AA \subseteq A$ and $SAS \subseteq A$

+ $A = \bar{A}$

k-ideals [Henriksen;1958]

What is an *k*-ideal ? How important is *k*-ideals ? What does "*k*" mean ?

M. Henriksen, *Ideals in semirings with commutative addition*, Amer. Math. Soc., notice 6(1958), 321.

542-183. Melvin Henriksen: Ideals in semirings with commutative addition.

Let S denote a semiring in the sense of the preceeding abstract. An ideal I of S is a nonempty subset I of S such that a, b ∈ I, and x ∈ S imply that a + b, xa, and ax are in I. A k-ideal I of S is an ideal of S such that x,x + y ∈ I imply that y ∈ S. A subset I of S is the kernel of a homomorphism iff I is a k-ideal. A ring without an identity element may contain ideals in the above sense that are not ideals in the ring-theoretic sense. The k-ideals of S (partially ordered by set inclusion) form a modular lattice, while (as is known) the lattice of ideals of S need not be modular. If S(·) has an identity element, then every proper k-ideal of S is contained in a maximal (proper) k-ideal. For background, see S. Bourne, (Proc. Nat. Acad. Sci. U. S. A. vol. 42 (1956) pp. 632-638). (Received December 13, 1957.)

S, T – semirings with absorbing zero

A function $\varphi: S \to T$ is called a **homomorphism** if

$$f(a+b) = f(a) + f(b) \text{ and } f(ab) = f(a)f(b)$$

for all $a, b \in S$

$$ker\varphi = \{a \in S \mid \varphi(a) = 0_T\}$$

$(S, +, \cdot)$-a additively comm. semiring

- A **k-ideal** I of S is an ideal of S such that $x, x+y \in I$ imply that $y \in I$.
- "A subset I of S is the kernel of a homomorphism iff I is a *k*-ideal."

EX: The ideal $2\mathbb{N}$ of the semiring $(\mathbb{N}, \max, \cdot)$ is not a *k*-ideal, since $\max\{1,2\} = 2 \in 2\mathbb{N}$ but $1 \notin 2\mathbb{N}$

A congruence relation on a semiring

A relation $\rho = \{(x, y) \in S \times S\}$ is an equivalence relation on a semiring S if the following conditions are satisfied:

- $(a, a) \in \rho$ for all $a \in S$;
- $(a, b) \in \rho \Rightarrow (b, a) \in \rho$ for all $a, b \in S$;
- $(a, b) \in \rho$ and $(b, c) \in \rho \Rightarrow (a, c) \in \rho$ for all $a, b, c \in S$.

An equivalent relation ρ is a congruence on a semiring S if for all $a, b, c, d \in S$,

$$(a, b), (c, d) \in \rho \text{ implies } (a + c, b + d), (ac, bd) \in \rho.$$

$$\Updownarrow$$

$$(a, b) \in \rho \text{ implies } (c + a, c + b), (a + c, b + c), (ca, cb), (ac, bc) \in \rho.$$

Bourne's relation (1951)

S. Bourne, The Jacobson radical of a semiring, Proc. Nat. Acad. Sci. 37(1951), 163-170.

THE JACOBSON RADICAL OF A SEMIRING

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Communicated by H. S. Vandiver, December 18, 1950

1. *Introduction.*—A semiring is a system consisting of a set S together with two binary operations, called addition and multiplication, which forms a semigroup relative to addition, a semigroup relative to multiplication, and the right and left distributive laws hold. This system was first introduced by Vandiver.[1] He also gave examples[2] of semirings which cannot be imbedded in a ring. Semirings arise naturally when we consider the set of endomorphisms of a commutative additive semigroup.[3]

Our purpose is to generalize the concept of the Jacobson radical of a ring[4] to arbitrary semirings. In section 2 we define the concept of an ideal in a semiring S and develop the corresponding homomorphism theorem for semirings. In section 3 we extend the definition of the Jacobson radical to arbitrary semirings, and in section 4 we obtain some properties of the Jacobson radical of a semiring. We conclude with a consideration, in section 5, of the Jacobson radical of matrix semiring S_n.

This paper has profited greatly from discussion with C. A. Rogers, a colleague of mine at the Institute.

2. *The Homomorphism Theorem.*—We shall assume that the additive semigroup of S is commutative and that S possesses a zero element. The latter assumption is not vital in the sense that if S lacked a zero element, we can easily adjoin one to S.

Definition 1: An ideal of S is a subset I of S containing zero such that if i_1 and i_2 are in I, then $i_1 + i_2$ is in I, and if i is in I, and s is any element of S, then is and si are in I.

We shall say that s_1 is equivalent to s_2 modulo the ideal I, if there exist elements i_1 and i_2 of the ideal I such that $s_1 + i_1 = s_2 + i_2$. This definition is a translation to the additive notation of one given by Dubreil[5] for a multiplicative semigroup. This relationship is obviously an equivalence.

Bourne's relation

$(S, +, \cdot)$-a additively comm. semiring with absorbing zero

I - an ideal of S and $s, t \in S$

$$s \sim t \Leftrightarrow s + i = t + j \text{ for some } i, j \in I$$
$$\Leftrightarrow s + I = t + I$$

Then $\sim$ is a congruence relation on S.

Pf: Clearly, $\sim$ is a equivalence relation on S.

Let $(a,b),(c,d) \in \sim$. Then $a + i_1 = b + j_1$ and $c + i_2 = d + j_2$ for some $i_1, i_2, j_1, j_2 \in I$. We have

$$(a + i_1) + (c + i_2) = (b + j_1) + (d + j_2)$$
$$(a + c) + i_1 + i_2 = (b + d) + j_1 + j_2$$
$$(a + i_1)(c + i_2) = (b + j_1)(d + j_2)$$
$$ac + i_1 c + a i_2 + i_1 i_2 = bd + j_1 d + b j_2 + j_1 j_2$$

Then $(a + c, b + d), (ac, bd) \in \sim$.

Now, $\sim$ is a congruence relation on S.

$S/I = \{[s]_\sim \mid s \in S\}$ the set of all cong. classes.

$[s]_\sim + [t]_\sim = [s+t]_\sim$

Clearly, $(S/I, +, \cdot)$ is a semiring. $[s]_\sim \cdot [t]_\sim = [st]_\sim$

Bourne's relation (1951)

S. Bourne, The Jacobson radical of a semiring, Proc. Nat. Acad. Sci. 37(1951), 163-170.

"A subset I of S is the kernel of a homomorphism iff I is a k-ideal."

Theorem: Let $\varphi: S \to T$ be a semiring hom. Then $\ker\varphi$ is a k-ideal of S.

Pf: Let $s \in S, a, b \in \ker\varphi$. We have

$$\varphi(a+b) = \varphi(a) + \varphi(b) = 0_T + 0_T = 0_T.$$
$$\varphi(as) = \varphi(a)\varphi(s) = 0_T\varphi(s) = 0_T.$$
$$\varphi(sa) = \varphi(s)\varphi(a) = \varphi(s)0_T = 0_T.$$

Assume that $s + a = b$. Then

$$0_T = \varphi(b) = \varphi(s+a) = \varphi(s) + \varphi(a) = \varphi(s) + 0_T = \varphi(s).$$

Therefore, $s \in \ker\varphi$.

Now, $\ker\varphi$ is a k-ideal of S.

Theorem: Let I be a k-ideal of S. The function $\varphi: S \to S/I$ defined by $\varphi(s) = [s]_\sim$ is a hom. and $I = \ker\varphi$.

Pf: $a \in I \Rightarrow a + 0 = 0 + a$
$\Rightarrow [a]_\sim = [0]_\sim$

$[a]_\sim = [0]_\sim \Rightarrow a + i = 0 + j = j \quad \exists i, j \in I$
$\Rightarrow a \in I, \quad (\because I \text{ is a } k-\text{ideal})$

Now, $a \in I$ iff $[a]_\sim = [0]_\sim$.

Therefore,

$$\ker\varphi = \{a \in S \mid \varphi(a) = [a]_\sim = [0]_\sim\}$$
$$= \{a \in S \mid a \in I\}$$
$$= I.$$

Additively inverse semirings

- An element a of a semiring S is called additively regular if $a = a + b + a$, $\exists b \in S$.
- If b is unique and $b = b + a + b$, then b is called the additively inverse of a.

M. K. Sen, M. Adhikari, On k-ideals of semirings, *International Journal of Mathematics and Mathematical Sciences*, 15 (1992), 347-350.

b is usually denoted by a'

Internat. J. Math. & Math. Sci.
VOL. 15 NO. 2 (1992) 347-350
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ON *k*-IDEALS OF SEMIRINGS

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(Received December 21, 1990)

- A semiring S is called additively regular if a is additively regular for all $a \in S$.
- A semiring S is called additively inverse if a has a' for all $a \in S$.

A ring congruence of a semiring

A relation $\rho = \{(x, y) \in S \times S\}$ is an equivalence relation on a semiring S if the following conditions are satisfied:

- $(a, a) \in \rho$ for all $a \in S$;
- $(a, b) \in \rho \Rightarrow (b, a) \in \rho$ for all $a, b \in S$;
- $(a, b) \in \rho$ and $(b, c) \in \rho \Rightarrow (a, c) \in \rho$ for all $a, b, c \in S$.

An equivalent relation ρ is a congruence on a semiring S if for all $a, b, c \in S$,

$$(a, b) \in \rho \text{ implies } (c + a, c + b), (a + c, b + c), (ca, cb), (ac, bc) \in \rho.$$

A congruence relation ρ on a semiring S is called a ring congruence if the quotient semiring S/ρ is a ring.

Full *k*-ideals of semirings

An element e of a semiring S is called an additively idempotent of S if $e + e = e$.

The set of all additively idempotent elements of a semiring S is

$$E^+(S) = \{x \in S \mid x + x = x\}.$$

A *k*-ideal A of a semiring S is a full *k*-ideal of S if $E^+(S) \subseteq A$.

Full k-ideals and ring congruences

Theorem A.

Let A be a full k-ideal of an additively inverse and commutative semiring S. Then the relation

$$\rho_A = \{(a,b) \in S \times S \mid a + b' \in A\}$$

is a ring congruence of S.

A is a full k-ideal of $S \Rightarrow S/\rho_A$ is a ring.

Theorem B.

Let ρ be a congruence relation on an additively inverse and commutative semiring S such that S/ρ is a ring. Then there exists a full k-ideal A of S such that $\rho = \rho_A$.

S/ρ is a ring $\Rightarrow \rho = \rho_A$ for some a full k-ideal A.

An n-ary groupoid

An n-ary groupoid is an algebra (S, f) such that $f: S^n \to S$ is an n-ary operation on S.

Notation Let $i, j, n \in \mathbb{N}$ be such that $1 \leq i < j \leq n$.

Let $x_1, x_2, x_3, \dots, x_n, x \in S$ and $A_1, A, A_3, \dots, A_n, A \subseteq S$.

Sequences	Representations
$x_i, x_{i+1}, \dots, x_j$	x_i^j
$x_1, x_2, \dots, x_j$ where $x_1 = x_2 = \cdots = x_j = x$	x^j
$A_i, A_{i+1}, \dots, A_j$	A_i^j
$A_1, A_2, \dots, A_j$ where $A_1 = A_2 = \cdots = A_j = A$	A^j

An *n*-ary semigroup

An n-ary groupoid (S, f) is an *n*-ary semigroup if for each $1 \leq i < j \leq n$,

$$f\left(x_1^{i-1}, f\left(x_i^{n+i-1}\right), x_{n+i}^{2n-1}\right) = f\left(x_1^{j-1}, f\left(x_j^{n+j-1}\right), x_{n+j}^{2n-1}\right)$$

for all $x_1^{2n-1} \in S$.

An *n*-ary semiring

An *n*-ary semiring is an algebra $(S,+,f)$ such that

- $(S,+)$ is a semigroup;
- (S,f) is an *n*-ary semigroup;
- for each $1 \leq i \leq n$,

$$f\left(x_1^{i-1}, a+b, x_{i+1}^n\right) = f\left(x_1^{i-1}, a, x_{i+1}^n\right) + f\left(x_1^{i-1}, b, x_{i+1}^n\right)$$

for all $x_1^n, a, b \in S$.

W. Dudek, On the divisibility theory in (*m*,*n*)-rings, *Demonstr. Math.*, 14 (1981), 19–32.

An *n*-ary semiring $(S,+,f)$ is an *n*-ary ring if $(S,+)$ is a commutative group.

Ideals and *k*-ideals of *n*-ary semirings

Let S be an *n*-ary semiring and $\emptyset \neq A \subseteq S$.

Ideals	Definitions
Ideal	$A + A \subseteq A$ $f(S^{i-1}, A, S^{n-i}) \subseteq A$ for each $1 \leq i \leq n$
k-ideal	$A + A \subseteq A$ $f(S^{i-1}, A, S^{n-i}) \subseteq A$ for each $1 \leq i \leq n$ $A = \bar{A}$

$$\bar{A} = \{x \in S \mid x + a \in A \text{ for some } a \in A\}$$

Full *k*-ideals of additively inverse *n*-ary semirings

An *n*-ary semiring $(S, +, f)$ is **additively inverse** if $(S, +)$ is an inverse semigroup.

An element e of an *n*-ary semiring S is called an **additively idempotent** of S if $e + e = e$.

The set of all additively idempotent elements of an *n*-ary semiring S is

$$E^+(S) = \{x \in S \mid x + x = x\}.$$

A *k*-ideal A of an *n*-ary semiring S is a **full *k*-ideal** of S if $E^+(S) \subseteq A$.

An *n*-ary ring congruence of an *n*-ary semiring

An equivalent relation ρ is a congruence on an *n*-ary semiring S if for all $a, b, c \in S$,

$$(a, b) \in \rho \text{ implies } (c + a, c + b), (a + c, b + c) \in \rho$$

and for each $1 \leq i \leq n$, $x_1, x_2, \ldots x_n \in S$,

$$\left(f\left(x_1^{i-1}, a, x_{i+1}^n\right), f\left(x_1^{i-1}, b, x_{i+1}^n\right)\right) \in \rho.$$

A congruence relation ρ on an *n*-ary semiring S is called an *n*-ary ring congruence if the quotient *n*-ary semiring S/ρ is an *n*-ary ring.

Full k-ideals and n-ary ring congruences

Theorem A.

$(S, +, f)$

Let A be a full k-ideal of an additively inverse and commutative n-ary semiring S. Then the relation

$g : A^m \to A$

$$\rho_A = \{(a, b) \in S \times S \mid a + b' \in A\}$$

is an n-ary ring congruence of S.

A is a full k-ideal of $S \Rightarrow S/\rho_A$ is an n-ary ring.

Theorem B.

Let ρ be a congruence relation on an additively inverse and commutative n-ary semiring S such that S/ρ is an n-ary ring. Then there exists a full k-ideal A of S such that $\rho = \rho_A$.

S/ρ is an n-ary ring $\Rightarrow \rho = \rho_A$ for some a full k-ideal A.

Full k-ideals and n-ary ring congruences

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On n-ary ring congruences of n-ary semirings

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Abstract: In universal algebra, it is well-known that if S is an algebraic structure, then the kind of algebraic structure of S/ρ is similar to S where ρ is a congruence relation on S. In this work, we study the notion of a full k-ideal A of an n-ary semiring S and construct a congruence relation ρ on S with respect to the full k-ideal A in order to make the quotient n-ary semiring S/ρ to be an n-ary ring. Moreover, the notion of an h-ideal of an n-ary semiring was studied and connections between an h-ideal and a k-ideal of an n-ary semiring were investigated.