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EINSTEIN GYROGROUP AS A B-LOOP*

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Using the Clifford algebra formalism, we give an algebraic proof that the open unit ball $\mathbb{B} = \{\mathbf{v} \in \mathbb{R}^n : \|\mathbf{v}\| < 1\}$ of $\mathbb{R}^n$ equipped with Einstein addition $\oplus_E$ forms a B-loop or, equivalently, a uniquely 2-divisible gyrocommutative gyrogroup. We obtain a compact formula for Einstein addition in terms of Möbius addition. We then give a characterization of associativity and commutativity of vectors in $\mathbb{B}$ with respect to Einstein addition.

Keywords: Einstein velocity addition, gyrogroup, B-loop.

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1. Introduction

Gyrogroup theory, introduced by Abraham A. Ungar, is related to various fields, including mathematical physics. For instance, the gyrogroup structure appears as an algebraic structure that encodes Einstein's velocity addition law [11,14]. It is also an algebraic structure that underlies the qubit density matrices, which play an important role in quantum mechanics [6,10]. For a connection to Thomas precession, see [15]. Of particular importance is the following composition law of Lorentz boosts,

$$L(\mathbf{u}) \circ L(\mathbf{v}) = L(\mathbf{u} \oplus_E \mathbf{v}) \circ \mathrm{Gyr}[\mathbf{u}, \mathbf{v}],$$

where $L(\mathbf{u})$ and $L(\mathbf{v})$ stand for Lorentz boosts parameterized by $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3_c$ and $\mathrm{Gyr}[\mathbf{u}, \mathbf{v}]$ is a rotation of spacetime coordinates induced by the Einstein gyroautomorphism generated by $\mathbf{u}$ and $\mathbf{v}$ [12, p. 448]. Connections between Einstein addition, Möbius addition, and hyperbolic geometry are described in [7,9]. For a connection to loops, see [5].

In [11], Einstein velocity addition, $\oplus_E$, on the set of relativistically admissible velocities, $\mathbb{R}^3_c = \{\mathbf{v} \in \mathbb{R}^3 : \|\mathbf{v}\| < c\}$, is given by the equation

$$\mathbf{u} \oplus_E \mathbf{v} = \frac{1}{1 + \langle \mathbf{u}, \mathbf{v} \rangle / c^2} \left\{ \mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}} \mathbf{v} + \frac{1}{c^2} \frac{\gamma_{\mathbf{u}}}{1 + \gamma_{\mathbf{u}}} \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{u} \right\},$$

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where c is a positive constant representing the speed of light in vacuum and $\gamma_{\mathbf{u}}$ is the Lorentz factor given by

$$\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1 - \|\mathbf{u}\|^2/c^2}}.$$

The system $(\mathbb{R}_c^3, \oplus_E)$ does not form a group since $\oplus_E$ is neither associative nor commutative. Nevertheless, $(\mathbb{R}_c^3, \oplus_E)$ is rich in structure and encodes a group-like structure, namely the gyrogroup structure. Ungar declared that $(\mathbb{R}_c^3, \oplus_E)$ forms a gyrocommutative gyrogroup, the so-called *Einstein gyrogroup*, where the gyrogroup axioms can be checked using computer algebra. It seems to us that no solid proof of this result is given in the literature. For this reason, we use the Clifford algebra formalism to prove this result. It turns out that *Einstein gyroautomorphisms*, also known as *Thomas gyrations*, can be expressed in a simple form using Clifford algebra operations.

Another example of a gyrogroup is the *Möbius gyrogroup*, which consists of the complex unit disk $\mathbb{D} = \{z \in \mathbb{C}\colon\ |z| < 1\}$ and Möbius addition

$$a \oplus_M b = \frac{a + b}{1 + \bar{a}b}, \qquad a, b \in \mathbb{D}.$$

In [13], the complex Möbius addition is extended to the Euclidean one,

$$\mathbf{u} \oplus_M \mathbf{v} = \frac{(1 + 2\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{v}\|^2)\mathbf{u} + (1 - \|\mathbf{u}\|^2)\mathbf{v}}{1 + 2\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{u}\|^2\|\mathbf{v}\|^2}, \qquad \mathbf{u}, \mathbf{v} \in \mathbb{B}.$$

Here, $\mathbb{B}$ denotes the open unit ball of $\mathbb{R}^n$,

$$\mathbb{B} = \{\mathbf{v} \in \mathbb{R}^n\colon\ \|\mathbf{v}\| < 1\}.$$

Because the formula for the Euclidean version of Möbius addition is very complicated, Lawson [8] and Ferreira and Ren [2] used the Clifford algebra formalism to study the Möbius gyrogroup and to simplify Möbius addition,

$$\mathbf{u} \oplus_M \mathbf{v} = (\mathbf{u} + \mathbf{v})(1 - \mathbf{u}\mathbf{v})^{-1}, \tag{1}$$

where the product and inverse on the right-hand side of Eq. (1) are performed in the Clifford algebra of *negative* Euclidean space.

With the compact formula (1) for Möbius addition in hand, we give an algebraic proof that the unit ball of $\mathbb{R}^n$ with Einstein addition does form a B-loop or a gyrocommutative gyrogroup with the uniquely 2-divisible property. As a consequence, we give a characterization of associativity and commutativity of the elements of Einstein gyrogroup $(\mathbb{B}, \oplus_E)$.

2. Preliminaries

Let $(G, \oplus)$ be a magma. Denote the group of automorphisms of G with respect to $\oplus$ by $\operatorname{Aut}(G, \oplus)$.

DEFINITION 1. ([12]) A magma $(G, \oplus)$ is a *gyrogroup* if its binary operation satisfies the following axioms.

(G1) There is an element $0 \in G$ such that $0 \oplus a = a$ for all $a \in G$.(left identity)

(G2) For each element $a \in G$, there is an element $b \in G$ such that $b \oplus a = 0$.(left inverse)

(G3) For all $a, b \in G$, there is an automorphism $\mathrm{gyr}[a, b] \in \mathrm{Aut}\,(G, \oplus)$ such that for all $c \in G$,

$$a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b]c. \qquad \text{(left gyroassociative law)}$$

(G4) For all $a, b \in G$, $\mathrm{gyr}[a, b] = \mathrm{gyr}[a \oplus b, b]$. (left loop property)

DEFINITION 2. ([12]) A gyrogroup $(G, \oplus)$ having the additional property that

$$a \oplus b = \mathrm{gyr}[a, b](b \oplus a) \qquad \text{(gyrocommutative law)}$$

for all $a, b \in G$ is called a *gyrocommutative gyrogroup*.

We remark that the axioms in Definition 1 imply the right counterparts. The map $\mathrm{gyr}[a, b]$ is called the *gyroautomorphism generated by a and b*. We refer the reader to [12] for a deep discussion of gyrogroups.

DEFINITION 3. A magma L has the *uniquely 2-divisible property* if the squaring map $x \mapsto x^2$ is a bijection from L to itself.

DEFINITION 4. A loop L has the *A_ℓ-property* if the left inner mapping

$$\ell(a, b) := L_{ab}^{-1} \circ L_a \circ L_b$$

defines an automorphism of L for all $a, b \in L$. Here, L_a denotes the left multiplication map by a defined by L_a: $x \mapsto ax$ for $x \in L$.

A loop L is called a *K-loop* or *Bruck loop* if every element of L has a unique inverse and L satisfies the left Bol identity (I) and the automorphic inverse property (II):

(I) $a(b(ac)) = (a(ba))c$

(II) $(ab)^{-1} = a^{-1}b^{-1}$

for all $a, b, c \in L$. A loop L is called a *B-loop* if it is a uniquely 2-divisible K-loop. In the literature, it is known that gyrogroups and left Bol loops with the A_ℓ-property are equivalent, and that uniquely 2-divisible gyrocommutative gyrogroups and B-loops are equivalent.

In order to prove that the unit ball of $\mathbb{R}^n$ with Einstein addition forms a uniquely 2-divisible gyrocommutative gyrogroup, we make use of the following theorem.

THEOREM 1. (Theorem 1, [1]) *Let $(G, \oplus)$ be a gyrogroup, X an arbitrary space, and ϕ: $X \to G$ a bijection between G and X. Then X endowed with the induced operation*

$$a \oplus_X b := \phi^{-1}(\phi(a) \oplus \phi(b))$$

for $a, b \in X$ becomes a gyrogroup.

PROPOSITION 1. *Let $(G, \oplus)$ be a gyrogroup. If $(G, \oplus)$ is gyrocommutative, then so is the induced gyrogroup $(X, \oplus_X)$. If $(G, \oplus)$ is uniquely 2-divisible, then so is $(X, \oplus_X)$.*

Proof: The proof of the first statement is straightforward. Let D_G and D_X denote the doubling maps of G and X, respectively. Assume that G is uniquely 2-divisible, that is, D_G is bijective. For all $x \in X$,

$$D_X(x) = x \oplus_X x = \phi^{-1}(\phi(x) \oplus \phi(x)) = \phi^{-1}(D_G(\phi(x))) = (\phi^{-1} \circ D_G \circ \phi)(x).$$

It follows that $D_X = \phi^{-1} \circ D_G \circ \phi$ and hence D_X is bijective, which proves that X is uniquely 2-divisible. □

3. Quadratic spaces and Clifford algebras

Let V be a vector space over a field $\mathbb{F}$ of characteristic different from 2. A *quadratic form* Q on V is a map $Q\colon V \to \mathbb{F}$ such that
(1) $Q(\lambda v) = \lambda^2 Q(v)$ for all $\lambda \in \mathbb{F}$, $v \in V$ and
(2) the map $B\colon V \times V \to \mathbb{F}$ defined by

$$B(u, v) = \frac{1}{2}\Big(Q(u + v) - Q(u) - Q(v)\Big)$$

is a symmetric bilinear form on V.
Note that any symmetric bilinear form B on V gives rise to a quadratic form Q by defining $Q(v) = B(v, v)$ for $v \in V$. A *quadratic space* is a vector space equipped with a quadratic form on which the associated bilinear form is nondegenerate. Let (V, Q) be a quadratic space with the corresponding bilinear form B. A basis $\{e_1, e_2, \ldots, e_n\}$ of V is *orthogonal* if $B(e_i, e_j) = 0$ for all $i \neq j$.

Let $\{e_1, e_2, \ldots, e_n\}$ be an orthogonal basis for (V, Q). The *Clifford algebra* of (V, Q), written $\mathrm{C}\ell_Q$, is a unital associative algebra over $\mathbb{F}$ with a basis

$$\{e_I\colon I = \emptyset \text{ or } I = \{1 \le i_1 < i_2 < \cdots < i_k \le n\}\},$$

where $e_\emptyset := 1$ and $e_I := e_{i_1} e_{i_2} \cdots e_{i_k}$ for $I = \{1 \le i_1 < i_2 < \cdots < i_k \le n\}$. From this a typical element of $\mathrm{C}\ell_Q$ is of the form $\sum_I \lambda_I e_I$, $\lambda_I \in \mathbb{F}$. Vector addition and scalar multiplication of $\mathrm{C}\ell_Q$ are defined pointwise, and multiplication is performed by using the distributive law without assuming commutativity subject to the defining relations

$$e_i^2 = Q(e_i)1 \quad \text{and} \quad e_i e_j = -e_j e_i$$

for $i \neq j$.

In $\mathrm{C}\ell_Q$, one has relations $v^2 = Q(v)1$ and $uv + vu = 2B(u, v)1$ for all $u, v \in V$. The base field $\mathbb{F}$ is embedded into $\mathrm{C}\ell_Q$ by the map $\lambda \mapsto \lambda 1$, and V is naturally embedded into $\mathrm{C}\ell_Q$ by inclusion. For a deep discussion of Clifford algebras, we refer the reader to [4].

There are three standard maps of a Clifford algebra. One is an involutive algebra automorphism, and the others are involutive algebra anti-automorphisms. Table 1 summarizes basic properties of such maps.

MAP	TYPE	ON V
reversion $\rho(a) = \tilde{a}$	anti-automorphism	id_V
grade involution $\tau(a) = \hat{a}$	automorphism	$-\mathrm{id}_V$
Clifford conjugation $\kappa(a) = \overline{a}$	anti-automorphism	$-\mathrm{id}_V$

Table 1. Three standard maps of $\mathrm{C}\ell_Q$.

The *Clifford* or *Lipschitz group* of $\mathrm{C}\ell_Q$, written $\Gamma(Q)$, is defined via the grade involution as

$$\Gamma(Q) = \{g \in \mathrm{C}\ell_Q^{\times}\colon\ \forall v \in V, \hat{g}vg^{-1} \in V\}.$$

In the finite-dimensional case, $\Gamma(Q)$ does form a subgroup of the group of units of $\mathrm{C}\ell_Q$. Further, the grade involution descends to a group automorphism of $\Gamma(Q)$, and the reversion and Clifford conjugation descend to group anti-automorphisms of $\Gamma(Q)$.

Let $\eta\colon \mathrm{C}\ell_Q \to \mathrm{C}\ell_Q$ be the map defined by

$$\eta(a) = a\overline{a}$$

for $a \in \mathrm{C}\ell_Q$. It can be proved that $\eta(g)$ belongs to $\mathbb{F}^{\times}1 := \{\lambda 1\colon\ \lambda \in \mathbb{F}^{\times}\}$ for all $g \in \Gamma(Q)$ and hence the restriction of η to $\Gamma(Q)$ is a group homomorphism.

Proposition 2. *The restriction of η to $\Gamma(Q)$ is a group homomorphism from $\Gamma(Q)$ to $\mathbb{F}^{\times}1$. Furthermore, η is multiplicative over the set of products of vectors in V in the sense that*

$$\eta(v_1v_2\cdots v_k) = \eta(v_1)\eta(v_2)\cdots\eta(v_k)$$

for all $v_1, v_2, \ldots, v_k \in V$.

Proof: This is because $\eta(g)$ and $\eta(v)$ are scalar multiples of unity for all $g \in \Gamma(Q)$ and $v \in V$. □

Invertibility of elements of the form $1 + uv$

In this subsection, we provide a necessary and sufficient condition for invertibility of elements of the form $1 + uv$, where u and v are vectors in a quadratic space. Let (V, Q) be a quadratic space with the corresponding bilinear form B. From now on, the term *vector* is reserved for the elements of V.

Lemma 1. *If u, v and w are vectors, then so are uvu and $uvw + wvu$.*

Proof: This follows from the fact that $uv + vu = 2B(u, v)1$ for all $u, v \in V$. □

Proposition 3. *If u and v are vectors, then either*
(1) *$1 + uv$ is a product of vectors or*
(2) *$1 + uv$ belongs to $\Gamma(Q)$ and $\eta(1 + uv) = 1$.*

Proof: Recall that if w is a nonisotropic vector, then w is invertible and $w^{-1} = w/Q(w)$ is again a vector. If u or v is nonisotropic, then $1+uv$ is a product of vectors. We may therefore assume that u and v are isotropic. If $B(u, v) \neq 0$, then $u + v$ is invertible and

$$1 + uv = (u + v + 2B(u, v)u)(u + v)^{-1}$$

is a product of vectors. If $B(u, v)=0$, then $\eta(1+uv)=1+2B(u, v)1+Q(u)Q(v)1=1$. By the lemma, $\tau(1+uv)w(1+uv)^{-1} = w + wvu + uvw + uvwvu$ belongs to V for all $w \in V$. Hence, $1 + uv \in \Gamma(Q)$. □

PROPOSITION 4. *For all* $u, v \in V$, $1 + uv \in \Gamma(Q)$ *if and only if* $\eta(1 + uv) \neq 0$.

Proof: ($\Rightarrow$) If $1 + uv \in \Gamma(Q)$, then $\eta(1 + uv) \in \mathbb{F}^{\times}1$. Hence, $\eta(1 + uv) \neq 0$.

($\Leftarrow$) Suppose that $\eta(1 + uv) \neq 0$. By Proposition 3, either $1 + uv$ already belongs to $\Gamma(Q)$ or $1 + uv$ is a product of vectors. In the latter case, $1 + uv = w_1 w_2 \cdots w_k$ for some $w_1, w_2, \ldots, w_k$ in V. Because $0 \neq \eta(1 + uv) = \eta(w_1 w_2 \cdots w_k) = \eta(w_1)\eta(w_2)\cdots\eta(w_k)$, none of $\eta(w_i)$ are zeros. Thus, $w_1, w_2, \ldots$, and w_k are all nonisotropic vectors and hence $1 + uv$ belongs to $\Gamma(Q)$. □

4. Negative Euclidean space

The *negative Euclidean space* consists of the underlying vector space $\mathbb{R}^n$ with a nondegenerate symmetric bilinear form

$$B(\mathbf{u}, \mathbf{v}) = -\langle \mathbf{u}, \mathbf{v}\rangle, \qquad \mathbf{u}, \mathbf{v} \in \mathbb{R}^n,$$

where $\langle \cdot, \cdot\rangle$ denotes the usual Euclidean inner product of $\mathbb{R}^n$. Its associated quadratic form is given by $Q(\mathbf{v}) = -\|\mathbf{v}\|^2$ for $\mathbf{v} \in \mathbb{R}^n$.

For convenience, let $C\ell_n$ denote the Clifford algebra of negative Euclidean space, let Γ_n denote the Clifford group of $C\ell_n$, and let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be the standard basis of $\mathbb{R}^n$. From now on, we identify elements of $\mathbb{R}1$ with real numbers, that is, $r1 \leftrightarrow r$ for $r \in \mathbb{R}$.

PROPOSITION 5. *In the Clifford algebra* $C\ell_n$, *the following properties hold.*

(1) $\mathbf{uv} + \mathbf{vu} = -2\langle \mathbf{u}, \mathbf{v}\rangle$ *for all* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.
(2) $\mathbf{v}^2 = -\|\mathbf{v}\|^2$ *for all* $\mathbf{v} \in \mathbb{R}^n$.
(3) $\mathbf{e}_i^2 = -1$, $\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i$ *for* $1 \leq i, j \leq n$ *and* $i \neq j$.
(4) $1 - \mathbf{uv} \in \Gamma_n$ *and* $(1 - \mathbf{uv})^{-1} = \dfrac{1 - \mathbf{vu}}{\eta(1 - \mathbf{uv})}$ *for all* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ *with* $\|\mathbf{u}\|\|\mathbf{v}\| \neq 1$.
(5) $\eta(\mathbf{w}(1 - \mathbf{uv})^{-1}) = \dfrac{\eta(\mathbf{w})}{\eta(1 - \mathbf{uv})}$ *for all* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ *with* $\|\mathbf{u}\|\|\mathbf{v}\| \neq 1$.

Proof: Items (1)–(3) follow from the defining relations in $C\ell_n$.

(4) The Cauchy–Schwarz inequality gives

$$\begin{aligned}\eta(1-\mathbf{uv}) &= 1+2\langle\mathbf{u},\mathbf{v}\rangle+\|\mathbf{u}\|^2\|\mathbf{v}\|^2\\ &\geq 1-2\|\mathbf{u}\|\|\mathbf{v}\|+\|\mathbf{u}\|^2\|\mathbf{v}\|^2\\ &= (1-\|\mathbf{u}\|\|\mathbf{v}\|)^2.\end{aligned}$$

It follows that if $\mathbf{u},\mathbf{v}\in\mathbb{R}^n$ with $\|\mathbf{u}\|\|\mathbf{v}\|\neq 1$, then $\eta(1-\mathbf{uv})>0$ and hence $1-\mathbf{uv}\in\Gamma_n$ by Proposition 4. Since $1-\mathbf{uv}\in\Gamma_n$, we have

$$(1-\mathbf{uv})^{-1}=\frac{\overline{1-\mathbf{uv}}}{\eta(1-\mathbf{uv})}=\frac{1-\mathbf{vu}}{\eta(1-\mathbf{uv})}.$$

(5) Let $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{R}^n$ with $\|\mathbf{u}\|\|\mathbf{v}\|\neq 1$. If $\mathbf{w}=\mathbf{0}$, equality holds trivially. We may therefore assume that $\mathbf{w}\neq\mathbf{0}$. Hence, $\mathbf{w}\in\Gamma_n$. By Item (4), $1-\mathbf{uv}\in\Gamma_n$ and so

$$\eta(\mathbf{w}(1-\mathbf{uv})^{-1})=\eta(\mathbf{w})\eta((1-\mathbf{uv})^{-1})=\frac{\eta(\mathbf{w})}{\eta(1-\mathbf{uv})}$$

since η is a group homomorphism of Γ_n. □

5. Möbius and Einstein gyrogroups on $\mathbb{R}^n$

Using relations $\mathbf{v}^2=-\|\mathbf{v}\|^2$ and $\mathbf{uv}+\mathbf{vu}=-2\langle\mathbf{u},\mathbf{v}\rangle$ in the Clifford algebra of negative Euclidean space, Lawson [8] verified that

$$(\mathbf{u}+\mathbf{v})(1-\mathbf{uv})^{-1}=\frac{(1+2\langle\mathbf{u},\mathbf{v}\rangle+\|\mathbf{v}\|^2)\mathbf{u}+(1-\|\mathbf{u}\|^2)\mathbf{v}}{1+2\langle\mathbf{u},\mathbf{v}\rangle+\|\mathbf{u}\|^2\|\mathbf{v}\|^2}=\mathbf{u}\oplus_M\mathbf{v}$$

for all $\mathbf{u},\mathbf{v}\in\mathbb{B}$. Hence, the Euclidean version of Möbius addition has a compact formula analogous to the complex case. He also dealt with the group of Möbius transformations of $\mathbb{R}^n$ that preserve the open unit ball to prove that $(\mathbb{B},\oplus_M)$ is indeed a B-loop.

In light of the proof of Proposition 5, $\eta(1-\mathbf{uv})\geq 0$ for all $\mathbf{u},\mathbf{v}\in\mathbb{R}^n$. Hence, the notation

$$|1-\mathbf{uv}|:=\sqrt{\eta(1-\mathbf{uv})}$$

is meaningful whenever $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$. Further, $|\mathbf{v}|=\|\mathbf{v}\|$ for all $\mathbf{v}\in\mathbb{R}^n$.

THEOREM 2. ([8]) *The Möbius loop on the open unit ball in $\mathbb{R}^n$ forms a B-loop whose operation is given in terms of the Clifford algebra $\mathrm{C}\ell_n$ by*

$$\mathbf{u}\oplus_M\mathbf{v}=(\mathbf{u}+\mathbf{v})(1-\mathbf{uv})^{-1}. \tag{2}$$

The left inner mappings are given by $\ell(\mathbf{u},\mathbf{v})\mathbf{w}=q\mathbf{w}q^{-1}$, where $q=\dfrac{1-\mathbf{uv}}{|1-\mathbf{uv}|}$.

Combining Eq. (2) with Proposition 5 (5) gives

$$\eta(\mathbf{u}\oplus_M\mathbf{v})=\frac{\eta(\mathbf{u}+\mathbf{v})}{\eta(1-\mathbf{uv})} \tag{3}$$

for all $\mathbf{u},\mathbf{v}\in\mathbb{B}$.

From now on, we work in the Clifford algebra of negative Euclidean space, $C\ell_n$. In light of Theorem 1 and Proposition 1, we express Einstein addition via Möbius addition to deduce that $(\mathbb{B}, \oplus_E)$ forms a uniquely 2-divisible gyrocommutative gyrogroup.

For each $\mathbf{v} \in \mathbb{B}$, set

$$r_{\mathbf{v}} = \frac{1}{1 + \sqrt{1 - \|\mathbf{v}\|^2}}. \tag{4}$$

Then

$$r_{\mathbf{v}} = \frac{1 - \sqrt{1 - \|\mathbf{v}\|^2}}{\|\mathbf{v}\|^2}$$

and

$$r_{\mathbf{v}} = \frac{1}{1 + \sqrt{1 + \mathbf{v}^2}}$$

in $C\ell_n$. According to the Lorentz factor normalized to $c = 1$, we have

$$r_{\mathbf{v}} = \frac{1}{1 + \gamma_{\mathbf{v}}^{-1}} = \frac{\gamma_{\mathbf{v}}}{1 + \gamma_{\mathbf{v}}}.$$

It follows that $0 < r_{\mathbf{v}} < 1$. In fact, $r_{\mathbf{v}}$ is a solution to the quadratic equation $\|\mathbf{v}\|^2 x^2 - 2x + 1 = 0$ in the variable x. Hence,

$$\frac{2r_{\mathbf{v}}}{1 - r_{\mathbf{v}}^2 \mathbf{v}^2} = 1. \tag{5}$$

Let Ψ be the map defined on $\mathbb{B}$ by

$$\Psi(\mathbf{v}) = r_{\mathbf{v}}\mathbf{v}, \qquad \mathbf{v} \in \mathbb{B}. \tag{6}$$

Since $0 < r_{\mathbf{v}} < 1$, we have $\|\Psi(\mathbf{v})\| = \|r_{\mathbf{v}}\mathbf{v}\| = r_{\mathbf{v}}\|\mathbf{v}\| < 1$. Hence, $\Psi(\mathbb{B}) \subseteq \mathbb{B}$.

Let Φ be the map defined on $\mathbb{B}$ by

$$\Phi(\mathbf{v}) = \mathbf{v} \oplus_M \mathbf{v}. \tag{7}$$

From Eq. (2), we have

$$\Phi(\mathbf{v}) = \frac{2\mathbf{v}}{1 - \mathbf{v}^2} = \frac{2\mathbf{v}}{1 + \|\mathbf{v}\|^2}.$$

The map Φ is called the *doubling map* and is of importance for the study of Möbius and Einstein gyrogroups, see for instance [7].

In the case $\|\mathbf{v}\| = 0$, $\mathbf{v} = \mathbf{0}$ and hence $\|\Phi(\mathbf{v})\| = \|\Phi(\mathbf{0})\| = \|\mathbf{0}\| = 0$. In the case $0 < \|\mathbf{v}\| < 1$,

$$\|\Phi(\mathbf{v})\| = \frac{2}{\frac{1}{\|\mathbf{v}\|} + \|\mathbf{v}\|} < 1$$

since $\dfrac{1}{\|\mathbf{v}\|} + \|\mathbf{v}\| > 2$. It follows that $\Phi(\mathbb{B}) \subseteq \mathbb{B}$.

PROPOSITION 6. *The maps Ψ and Φ are bijections from $\mathbb{B}$ to itself and are inverses of each other.*

Proof: Let $\mathbf{v} \in \mathbb{B}$. Since $1 - \|\Phi(\mathbf{v})\|^2 = 1 - \dfrac{4\|\mathbf{v}\|^2}{(1+\|\mathbf{v}\|^2)^2} = \left(\dfrac{1-\|\mathbf{v}\|^2}{1+\|\mathbf{v}\|^2}\right)^2$, we have

$$1 + \sqrt{1 - \|\Phi(\mathbf{v})\|^2} = 1 + \frac{1-\|\mathbf{v}\|^2}{1+\|\mathbf{v}\|^2} = \frac{2}{1+\|\mathbf{v}\|^2} = \frac{2}{1-\mathbf{v}^2}.$$

It follows that $(\Psi \circ \Phi)(\mathbf{v}) = \Psi(\Phi(\mathbf{v})) = r_{\Phi(\mathbf{v})}\Phi(\mathbf{v}) = \dfrac{1}{1+\sqrt{1+\Phi(\mathbf{v})^2}}\dfrac{2\mathbf{v}}{1-\mathbf{v}^2} = \mathbf{v}.$

From Eq. (5), we have

$$(\Phi \circ \Psi)(\mathbf{v}) = \Phi(\Psi(\mathbf{v})) = \frac{2\Psi(\mathbf{v})}{1-\Psi(\mathbf{v})^2} = \frac{2r_{\mathbf{v}}}{1-r_{\mathbf{v}}^2\mathbf{v}^2}\mathbf{v} = \mathbf{v}.$$

This proves $\Psi \circ \Phi = \mathrm{id}_{\mathbb{B}}$ and $\Phi \circ \Psi = \mathrm{id}_{\mathbb{B}}$. Hence, Φ and Ψ are bijections, $\Phi^{-1} = \Psi$, and $\Psi^{-1} = \Phi$. $\square$

PROPOSITION 7. *The unit ball $\mathbb{B}$ with the induced operation*

$$\mathbf{u} \oplus_{\mathbb{B}} \mathbf{v} = \Psi^{-1}(\Psi(\mathbf{u}) \oplus_M \Psi(\mathbf{v})), \qquad \mathbf{u}, \mathbf{v} \in \mathbb{B},$$

forms a uniquely 2-divisible gyrocommutative gyrogroup.

Proof: The proposition follows from Theorem 1 and Proposition 1 applied to $(\mathbb{B}, \oplus_M)$ and Ψ. $\square$

In fact, the induced addition $\oplus_{\mathbb{B}}$ is nothing but Einstein addition, as shown in the following theorem.

THEOREM 3. *For all $\mathbf{u}, \mathbf{v} \in \mathbb{B}$,*

$$\mathbf{u} \oplus_{\mathbb{B}} \mathbf{v} = \mathbf{u} \oplus_E \mathbf{v}.$$

In particular, $(\mathbb{B}, \oplus_E)$ forms a uniquely 2-divisible gyrocommutative gyrogroup. In terms of the Clifford algebra $\mathrm{C}\ell_n$, Einstein addition can be rewritten as

$$\mathbf{u} \oplus_E \mathbf{v} = 2(r_{\mathbf{u}}\mathbf{u} \oplus_M r_{\mathbf{v}}\mathbf{v})\left(1 - (r_{\mathbf{u}}\mathbf{u} \oplus_M r_{\mathbf{v}}\mathbf{v})^2\right)^{-1} \tag{8}$$

and the Einstein gyroautomorphisms are given by

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} = q\mathbf{w}q^{-1}, \qquad q = \frac{1 - r_{\mathbf{u}}r_{\mathbf{v}}\mathbf{u}\mathbf{v}}{|1 - r_{\mathbf{u}}r_{\mathbf{v}}\mathbf{u}\mathbf{v}|},$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{B}$.

Proof: Since $\Psi^{-1} = \Phi$, we have

$$\mathbf{u} \oplus_{\mathbb{B}} \mathbf{v} = \Phi(\Psi(\mathbf{u}) \oplus_M \Psi(\mathbf{v})) = \frac{2}{1 - [\Psi(\mathbf{u}) \oplus_M \Psi(\mathbf{v})]^2}[\Psi(\mathbf{u}) \oplus_M \Psi(\mathbf{v})].$$

Note that $\eta(\Psi(\mathbf{u}) \oplus_M \Psi(\mathbf{v})) = [\Psi(\mathbf{u}) \oplus_M \Psi(\mathbf{v})]\overline{\Psi(\mathbf{u}) \oplus_M \Psi(\mathbf{v})} = -[\Psi(\mathbf{u}) \oplus_M \Psi(\mathbf{v})]^2$ since $\Psi(\mathbf{u}) \oplus_M \Psi(\mathbf{v}) \in \mathbb{R}^n$. Eqs. (2) and (3) and Proposition 5 together imply

$$\begin{aligned}
\mathbf{u} \oplus_{\mathbb{B}} \mathbf{v} &= \frac{2}{1 + \eta(\Psi(\mathbf{u}) \oplus_M \Psi(\mathbf{v}))}[\Psi(\mathbf{u}) \oplus_M \Psi(\mathbf{v})] \\
&= \frac{2}{1 + \dfrac{\eta(\Psi(\mathbf{u}) + \Psi(\mathbf{v}))}{\eta(1 - \Psi(\mathbf{u})\Psi(\mathbf{v}))}}[\Psi(\mathbf{u}) + \Psi(\mathbf{v})][1 - \Psi(\mathbf{u})\Psi(\mathbf{v})]^{-1} \\
&= \frac{2[\Psi(\mathbf{u}) + \Psi(\mathbf{v})][1 - \Psi(\mathbf{v})\Psi(\mathbf{u})]}{\eta(\Psi(\mathbf{u}) + \Psi(\mathbf{v})) + \eta(1 - \Psi(\mathbf{u})\Psi(\mathbf{v}))}.
\end{aligned} \tag{9}$$

Since $1 - r_{\mathbf{u}}^2\mathbf{u}^2 - r_{\mathbf{v}}^2\mathbf{v}^2 + r_{\mathbf{u}}^2\mathbf{u}^2 r_{\mathbf{v}}^2\mathbf{v}^2 = (1 - r_{\mathbf{u}}^2\mathbf{u}^2)(1 - r_{\mathbf{v}}^2\mathbf{v}^2) = (2r_{\mathbf{u}})(2r_{\mathbf{v}}) = 4r_{\mathbf{u}}r_{\mathbf{v}}$, we have

$$\begin{aligned}
\eta(\Psi(\mathbf{u}) + \Psi(\mathbf{v})) + \eta(1 - \Psi(\mathbf{u})\Psi(\mathbf{v})) &= 1 - r_{\mathbf{u}}^2\mathbf{u}^2 - r_{\mathbf{v}}^2\mathbf{v}^2 + r_{\mathbf{u}}^2\mathbf{u}^2 r_{\mathbf{v}}^2\mathbf{v}^2 + 4r_{\mathbf{u}}r_{\mathbf{v}}\langle \mathbf{u}, \mathbf{v}\rangle \\
&= 4r_{\mathbf{u}}r_{\mathbf{v}}(1 + \langle \mathbf{u}, \mathbf{v}\rangle).
\end{aligned} \tag{10}$$

We also have

$$\begin{aligned}
&\frac{1}{2r_{\mathbf{u}}r_{\mathbf{v}}}[\Psi(\mathbf{u}) + \Psi(\mathbf{v})][1 - \Psi(\mathbf{v})\Psi(\mathbf{u})] \\
&= \frac{\mathbf{u}}{2r_{\mathbf{v}}} - \frac{r_{\mathbf{u}}}{2}\mathbf{u}\mathbf{v}\mathbf{u} + \frac{\mathbf{v}}{2r_{\mathbf{u}}} - \frac{r_{\mathbf{v}}}{2}\mathbf{v}^2\mathbf{u} \\
&= \frac{1}{2}\left(\frac{1}{r_{\mathbf{v}}} - r_{\mathbf{v}}\mathbf{v}^2\right)\mathbf{u} - \frac{r_{\mathbf{u}}}{2}(\mathbf{u}\mathbf{v} + \mathbf{v}\mathbf{u})\mathbf{u} + \frac{1}{2}\left(r_{\mathbf{u}}\mathbf{u}^2 + \frac{1}{r_{\mathbf{u}}}\right)\mathbf{v} \\
&= \mathbf{u} + \frac{\gamma_{\mathbf{u}}}{1 + \gamma_{\mathbf{u}}}\langle \mathbf{u}, \mathbf{v}\rangle\mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v}.
\end{aligned} \tag{11}$$

We obtain the third equation of (11) because $\dfrac{1}{r_{\mathbf{w}}} = 1 + \sqrt{1 - \|\mathbf{w}\|^2} = 1 + \dfrac{1}{\gamma_{\mathbf{w}}}$ and

$$r_{\mathbf{w}}\mathbf{w}^2 = \frac{1 - \sqrt{1 - \|\mathbf{w}\|^2}}{\|\mathbf{w}\|^2}(-\|\mathbf{w}\|^2) = \sqrt{1 - \|\mathbf{w}\|^2} - 1 = \frac{1}{\gamma_{\mathbf{w}}} - 1$$

for all $\mathbf{w} \in \mathbb{B}$. Combining Eqs. (9)–(11) gives

$$\begin{aligned}
\mathbf{u} \oplus_{\mathbb{B}} \mathbf{v} &= \frac{4r_{\mathbf{u}}r_{\mathbf{v}}\left\{\mathbf{u} + \dfrac{\gamma_{\mathbf{u}}}{1 + \gamma_{\mathbf{u}}}\langle \mathbf{u}, \mathbf{v}\rangle\mathbf{u} + \dfrac{1}{\gamma_{\mathbf{u}}}\mathbf{v}\right\}}{4r_{\mathbf{u}}r_{\mathbf{v}}(1 + \langle \mathbf{u}, \mathbf{v}\rangle)} \\
&= \frac{1}{1 + \langle \mathbf{u}, \mathbf{v}\rangle}\left\{\mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v} + \frac{\gamma_{\mathbf{u}}}{1 + \gamma_{\mathbf{u}}}\langle \mathbf{u}, \mathbf{v}\rangle\mathbf{u}\right\} \\
&= \mathbf{u} \oplus_E \mathbf{v}.
\end{aligned}$$

The second part of the theorem follows from the result that

$$\operatorname{gyr}_E[\mathbf{u}, \mathbf{v}] = \ell(\Phi(\mathbf{u}), \Phi(\mathbf{v}))$$

and Theorem 2. □

Eq. (8) shows a close relationship between elements of Einstein and Möbius gyrogroups. See also [12, Eq. (6.297)] and [1, Proposition 6]. In terms of *Einstein scalar multiplication* [12, p. 218], given by

$$r \otimes_E \mathbf{v} = \frac{(1+\|\mathbf{v}\|)^r - (1-\|\mathbf{v}\|)^r}{(1+\|\mathbf{v}\|)^r + (1-\|\mathbf{v}\|)^r} \frac{\mathbf{v}}{\|\mathbf{v}\|}, \qquad r \in \mathbb{R}, \;\; \mathbf{0} \neq \mathbf{v} \in \mathbb{B}, \tag{12}$$

Eqs. (6) and (7) can be rewritten as

$$\Psi(\mathbf{v}) = \frac{1}{2} \otimes_E \mathbf{v} \quad \text{and} \quad \Phi(\mathbf{v}) = 2 \otimes_E \mathbf{v},$$

which reflects the fact that Ψ and Φ are inverses of each other.

Although the result that Einstein addition can be expressed via Möbius addition is known, see Friedman and Scarr [3, Eq. (2.13)], we obtain these results using a *different* technique. In fact, Friedman and Scarr obtained the result using the *principle of special relativity*, whereas we use an algebraic approach.

We end this section with the following characterization of associativity and commutativity of the elements of Einstein gyrogroup $(\mathbb{B}, \oplus_E)$.

THEOREM 4. *For all* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{B}$,

$$\mathbf{u} \oplus_E (\mathbf{v} \oplus_E \mathbf{w}) = (\mathbf{u} \oplus_E \mathbf{v}) \oplus_E \mathbf{w}$$

if and only if $\langle \mathbf{u}, \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle = 0$ *or* $\mathbf{u} \| \mathbf{v}$.

Proof: ($\Rightarrow$) Suppose that $\mathbf{u} \oplus_E (\mathbf{v} \oplus_E \mathbf{w}) = (\mathbf{u} \oplus_E \mathbf{v}) \oplus_E \mathbf{w}$. Since $\Psi\colon (\mathbb{B}, \oplus_E) \to (\mathbb{B}, \oplus_M)$ is a gyrogroup isomorphism, we have $\Psi(\mathbf{u}) \oplus_M (\Psi(\mathbf{v}) \oplus_M \Psi(\mathbf{w})) = (\Psi(\mathbf{u}) \oplus_M \Psi(\mathbf{v})) \oplus_M \Psi(\mathbf{w})$. By Lemma 10 of [2], $\langle \Psi(\mathbf{u}), \Psi(\mathbf{w}) \rangle = \langle \Psi(\mathbf{v}), \Psi(\mathbf{w}) \rangle = 0$ or $\Psi(\mathbf{u}) \| \Psi(\mathbf{v})$. By Eq. (6), $\langle r_{\mathbf{u}}\mathbf{u}, r_{\mathbf{w}}\mathbf{w} \rangle = 0 = \langle r_{\mathbf{v}}\mathbf{v}, r_{\mathbf{w}}\mathbf{w} \rangle$ or $r_{\mathbf{u}}\mathbf{u} \| r_{\mathbf{v}}\mathbf{v}$, which implies the desired statement.

($\Leftarrow$) If $\langle \mathbf{u}, \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle = 0$, then $\langle \Psi(\mathbf{u}), \Psi(\mathbf{w}) \rangle = \langle r_{\mathbf{u}}\mathbf{u}, r_{\mathbf{w}}\mathbf{w} \rangle = 0$. Similarly, $\langle \Psi(\mathbf{v}), \Psi(\mathbf{w}) \rangle = 0$. Hence, $\Psi(\mathbf{u}) \oplus_M (\Psi(\mathbf{v}) \oplus_M \Psi(\mathbf{w})) = (\Psi(\mathbf{u}) \oplus_M \Psi(\mathbf{v})) \oplus_M \Psi(\mathbf{w})$. Applying Φ to both sides of the equation gives $\mathbf{u} \oplus_E (\mathbf{v} \oplus_E \mathbf{w}) = (\mathbf{u} \oplus_E \mathbf{v}) \oplus_E \mathbf{w}$ since $\Phi = \Psi^{-1}$ and Φ preserves the operations. If $\mathbf{u} \| \mathbf{v}$, then $\Psi(\mathbf{u}) \| \Psi(\mathbf{v})$ and so the same reasoning applies. □

THEOREM 5. *For all* $\mathbf{u}, \mathbf{v} \in \mathbb{B}$,

$$\mathbf{u} \oplus_E \mathbf{v} = \mathbf{v} \oplus_E \mathbf{u}$$

if and only if $\mathbf{u} \| \mathbf{v}$.

Proof: If $\mathbf{u} \oplus_E \mathbf{v} = \mathbf{v} \oplus_E \mathbf{u}$, then $\Psi(\mathbf{u}) \oplus_M \Psi(\mathbf{v}) = \Psi(\mathbf{v}) \oplus_M \Psi(\mathbf{u})$. By Lemma 11 of [2], $\Psi(\mathbf{u}) \| \Psi(\mathbf{v})$. Hence, $\mathbf{u} \| \mathbf{v}$. Conversely, if $\mathbf{u} \| \mathbf{v}$, then $\Psi(\mathbf{u}) \| \Psi(\mathbf{v})$, which implies $\Psi(\mathbf{u}) \oplus_M \Psi(\mathbf{v}) = \Psi(\mathbf{v}) \oplus_M \Psi(\mathbf{u})$. Applying Φ to both sides of the equation gives $\mathbf{u} \oplus_E \mathbf{v} = \mathbf{v} \oplus_E \mathbf{u}$. □

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